# Laplace transform, regularized deconvolution : designing virtual temperature sensors

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## Definitions and objectives

A **virtual sensing** system uses:

- information available from other measurements and process parameters
  → estimate of the quantity of interest. [Wikipedia]

(This estimation) uses **mathematical models** (...)
which use **other physical sensor readings (...)** [www.intellidynamics.net]

**Present application:** ***Design of virtual thermal sensors***

for

- estimating temperature or heat flux or rate of heat flow or heat source

from

- temperature sensors **located at points different** from the points where either the temperature or heat flux is looked looked for

with prior knowledge

- of a corresponding mathematical model → ***convolutive structure*** *of adhoc models*

## *Gensesis of this work*

Personal/research team background:

1) **Parameter estimation for** thermophysical property characterization
2) **Function estimation:** inverse heat conduction problems (heat flux, ill-posed)

## *Scope of the talk*

Topic 1) Introduction to the world of time convolutions

→ in 0D and 1D cases, interest of Laplace transformation for transient diffusion problems

→ introduction of the impulse response in a **model identification** inverse problem:
time domain form of a convolution product

Topic 2) Laplace transformation useful in real 3D word:
conduction, forced convection, linearized radiation)

…… if some assumptions are verified

→ **convolutive models** (with their nice properties)

Applications to conjugated heat transfer :forced convection (fluid) + conduction (solid):
Thick channel – Plate fin heat exchanger

# Outline

1. **Laplace transform and Linear Ordinary Differential Equations with Time Independent coefficients:** properties, transfer function and convolution product – S4
2. **Laplace transform and 0 D heat transfer:** thermal impedance – S7
3. **Laplace transform and 1D heat transfer:** Thermal quadrupoles, impedance, transmittance, admittance and calculation of their time versions – S8
4. **Practical calculation of a convolution product and experimental deconvolution** – S12
5. **Laplace transform and 3 D heat transfer** – S16
6. **Laplace transform and steady state transfer functions** – S22
7. **Application to conjugated transfer in heat exchangers** – S24
   Thick channel – Plate fin heat exchanger: characterization and fouling detection
8. **Conclusions and perspectives** – S36

## 1. Laplace transform and Linear Ordinary Differential Equations with Time Independent coefficients (LTI)

$$\begin{cases} \dfrac{dy}{dt} + a\,y = b\,u \;;\; a \text{ and } b: \text{ constants} \\ y\,(t=0) = y_0 \end{cases}$$

LAPLACE TRANSFORMATION : $\bar{y}\,(p) = \mathcal{L}\left[\,y\,(t)\,\right] = \int_0^\infty \exp\,(-p\,t)\;y\,(t)\;dt$

Laplace parameter

**Property 1 :**

$$\mathcal{L}\left[\frac{dy}{dt}\right] = p\,\bar{y} - y_0$$

$$\bar{y}\,(p) = \frac{b}{p+a}\,\bar{u}\,(p) + \frac{1}{p+a}\,y_0 = \bar{y}_{forced}\,(p) + \bar{y}_{relax}\,(p)$$

$$\bar{y}_{forced}\,(p) = \bar{H}\,(p)\;\bar{u}\,(p)$$

Transfer function

$$\bar{u}\,(p) \rightarrow \boxed{\bar{H}\,(p)} \rightarrow \bar{y}_{forced}\,(p)$$

$$\bar{y}_{forced}\,(p) = \underset{\downarrow}{\bar{H}}\,(p)\,\bar{u}\;(p)$$

**Transfer function**

$$\bar{u}\;(p) \rightarrow \boxed{\bar{H}\,(p)} \rightarrow \bar{y}_{forced}\;(p)$$

**Property 2 :** $\mathcal{L}^{-1}\left[\bar{H}\,(p)\,\bar{u}\;(p)\right] = H\,(t) * u\;(t) \equiv \int_0^t H\,(t - t')\,u\,(t')\,\mathrm{d}t' = \int_0^t H\,(t')\,u\,(t - t')\,\mathrm{d}t'$

**convolution product**

$$\underset{\downarrow}{y}_{forced}\;(t) = \underset{\downarrow}{H}\,(t) * \underset{\downarrow}{u}\;(t) = \int_0^t H\,(t - t')\,u\,(t')\,\mathrm{d}t'$$

**Output**
**Response**
**(consequence)**

**Impulse**
**response**

**Input**
**Excitation**
**(cause)**

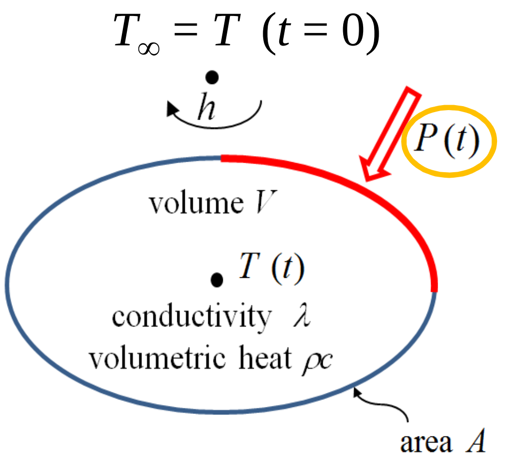
Here, ***mathematical problem*** :

Analytical expression of impulse response: $H\;(t) = b\,\exp(-\,a\;t)$

Analytical expression of relaxation solution: $y_{relax}\;(t) = y_0\;\exp\,(-a\,t)$

# 2. Laplace transform and 0 D heat transfer

Corresponding ***heat transfer problem***: lumped body (0 D) approximation



**Lumped body approximation :**

$Bi\ = h\,d\,/\,(2\,\lambda) << 1$ with $d = 6V\,/\,A$

characteristic time of $P\,(t) << \rho c\ d^2\,/\,\lambda$

Excitation : $u$ (t) = $P$ ($t$) (units: watts) starts at $t$ = $0^+$

Response :

$$y\,(t) = \theta_{forced}(t) = T\ (t)\ \text{-}\ T\ (t=0) \quad \text{(units: Kelvin)}$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} - \frac{1}{\tau}\theta = \frac{1}{\rho c V} P\,(t)$$

with $\theta\ (0) = \theta_0 = 0$ and $\tau = \dfrac{\rho c V}{h\,A}$

$$a = 1/\tau \quad (\text{units}: \text{s}^{-1}) \quad ; \quad b = 1/(\rho c V) \quad (\text{units}: \text{K/J})$$

Here $H$ (.) = thermal impedance, noted $Z$ (.)
(units: Kelvin/Joule)

$$\bar{Z}\ (p) = \frac{1}{\rho c V}\ \frac{b}{p - 1/\tau} \quad \Leftrightarrow \quad Z\ (t) = \frac{1}{\rho c}\ \frac{1}{V} \exp(\text{-}\,t\,/\,\tau)$$

**Transfert function**
**Operational impedance**

**Impulse response**
**= Time impedance**

$$\bar{P}\ (p) \rightarrow \boxed{\bar{Z}\,(p)} \rightarrow \bar{\theta}_{forced}\ (p)$$

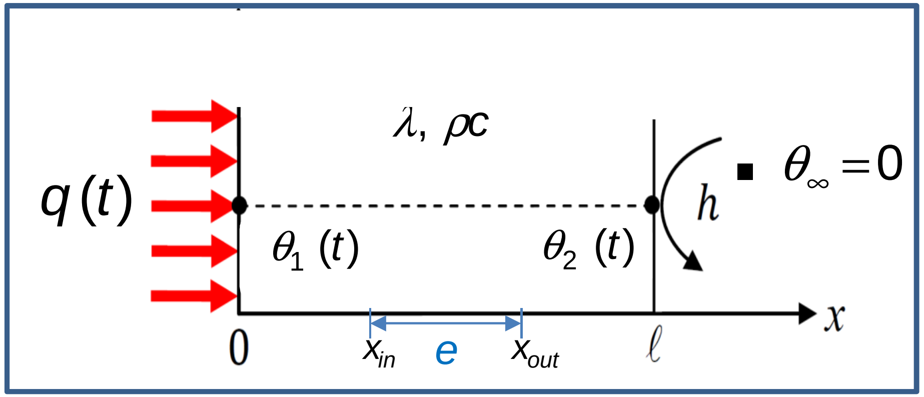
# 3. Laplace transform and 1 D heat transfer

Heat equation (1) $\left\{ \frac{\partial^2\theta}{\partial x^2} = \frac{1}{a}\frac{\partial\theta}{\partial t} \right.$

Initial Condition (2) $\left\{ \theta = 0 \quad \text{at} \quad t = 0 \quad \text{for} \quad 0 \le x \le \ell \right.$

Boundary Conditions (3) $\left\{ \begin{array}{l} \varphi = -\lambda \frac{\partial\theta}{\partial t} = q(t) \quad \text{at} \;\; x = 0 \quad \text{for} \quad t > 0\,; \\ \varphi = -\lambda \frac{\partial\theta}{\partial t} = h\,\theta \quad \text{at} \;\; x = \ell \quad \text{for} \quad t > 0 \end{array} \right.$

What happens in 1 D transient heat conduction ?



Use of the Thermal Quadrupole Method [1, 2, 3]

Laplace transform $\quad \overline{\psi}\ (x,\, p) \equiv \int_0^t \psi(x,\, t)\ \exp\,(-\,p\,t)\ dt \quad \text{for} \quad \psi = \theta \quad \text{or} \quad \varphi$

Temperature variation

Heat flux

## Fora sublayer of **any** thickness $e = x_2 - x_1$ :

$$(1)\ \text{and}(2) \Leftrightarrow \begin{bmatrix} \overline{\theta}\,(x_{in}, p) \\ \overline{\varphi}_1\,(x_{in}, p) \end{bmatrix} = \underbrace{\begin{bmatrix} A_e\,(p) & B_e\,(p) \\ C_e\,(p) & D_e\,(p) \end{bmatrix}}_{\textbf{Quadrupolar matrix}} \begin{bmatrix} \overline{\theta}\,(x_{out}, p) \\ \overline{\varphi}_2\,(x_{out}, p) \end{bmatrix}$$

(4 terminal port network in electricity)

with
$$\left\{ \begin{array}{c} A_e = D_e = \cosh\,(e\sqrt{p/a}\,)\ ;\ B_e = \dfrac{1}{\lambda\sqrt{p/a}}\ \sinh\,(e\sqrt{p/a}\,) \\ C_e = \lambda\sqrt{p/a}\ \sinh\,(e\sqrt{p/a}\,) \\ \text{where}\ \ a = \lambda\,/\,\rho c\ \ \text{and}\ \ e = x_{out} - x_{in} \ge 0 \end{array} \right.$$

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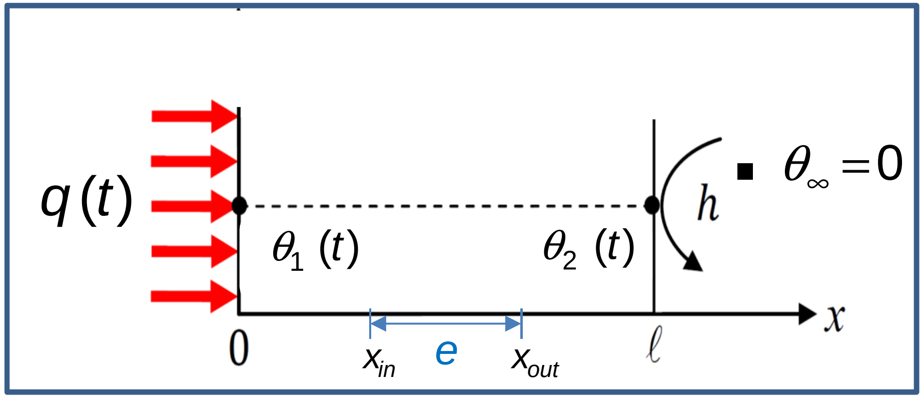
REFERENCES
[1] H.S. Carslaw & J.C. Jaeger,Conduction of Heat in Solids, Oxford U. Press, 1947
[2] L. A. Pipes,Matrix analysis of heat transfer problems, Journal of the Franklin Institute, vol. 263, n° 3, pp. 195-205, 1957
[3] D. Maillet, S. André, J.C. Batsale, A. Degiovanni, C. Moyne, Thermal Quadrupoles – Solving the heat equation through integral transforms, Wiley, 2000

$$\begin{bmatrix} \bar\theta(x_{in},p) \\ \bar\varphi_1(x_{in},p) \end{bmatrix} = \begin{bmatrix} A_e(p) & B_e(p) \\ C_e(p) & D_e(p) \end{bmatrix} \begin{bmatrix} \bar\theta(x_{out},p) \\ \bar\varphi_2(x_{out},p) \end{bmatrix}$$



Boundary Conditions (3)

$$\begin{cases} \varphi = -\lambda \dfrac{\partial\theta}{\partial t} = q(t) \quad \text{at } x = 0 \quad \text{for} \quad t > 0\,; \\ \varphi = -\lambda \dfrac{\partial\theta}{\partial t} = h\ \theta \quad \text{at } x = \ell \quad \text{for} \quad t > 0 \end{cases}$$

Here : $x_{in} = 0$ and $x_{out} = \ell \Rightarrow e = \ell$

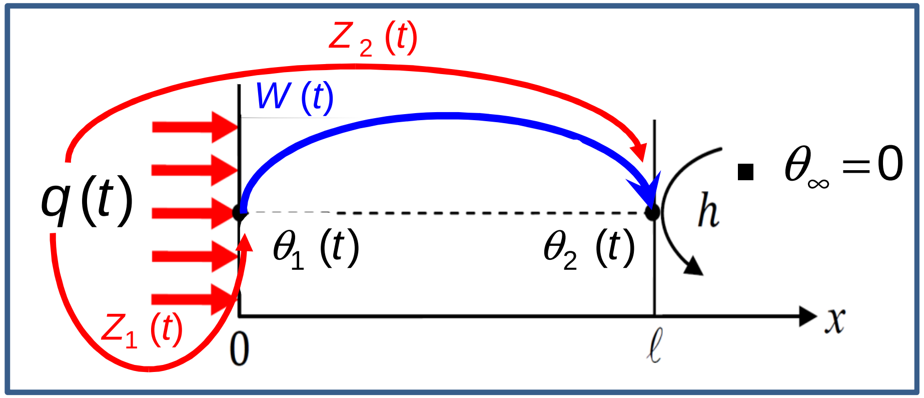
$$(3a) \Rightarrow \begin{bmatrix} \overset{\text{unknown}}{\overset{\downarrow}{\bar\theta_1(p)}} \\ \underset{\text{data}}{\underset{\uparrow}{\bar q(p)}} \end{bmatrix} = \underbrace{\begin{bmatrix} A_\ell(p) & B_\ell(p) \\ C_\ell(p) & D_\ell(p) \end{bmatrix}}_{\text{QP matrix}} \begin{bmatrix} \overset{\text{unknown}}{\overset{\downarrow}{\bar\theta_2(p)}} \\ \underset{\text{unknown}}{\underset{\uparrow}{\bar\varphi_2(p)}} \end{bmatrix} \text{ and (3b)} \Rightarrow \begin{bmatrix} \bar\theta_2(p) \\ \bar\varphi_2(p) \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ h & 1 \end{bmatrix}}_{\text{QP matrix}} \begin{bmatrix} \bar\theta_2(p) \\ 0 \end{bmatrix}$$

3 equations
3 unknowns

Product of QP matrices → $\begin{bmatrix} \bar\theta_1(p) \\ \bar q(p) \end{bmatrix} = \begin{bmatrix} A(p) & B(p) \\ C(p) & D(p) \end{bmatrix} \begin{bmatrix} \bar\theta_2(p) \\ 0 \end{bmatrix}$

Solution in Laplace domain → $\bar\theta_1 = \dfrac{A}{C}\ \bar q \quad ; \quad \bar\theta_2 = \dfrac{1}{C}\ \bar q \quad ; \quad \bar\varphi_2 = \dfrac{h}{C}\ \bar q$

- Return to the time domain: inversion of Laplace transform = ill-posed problem
- Simple cases: Analytical solutions in simple cases: 1) Laplace transform tables 2) Rational fractions (zeros/poles)
- General case:
  - Broomwich integral (involved technique)
  - Numerical inversion: 1) Stehfest's algorithm, 2) through Fourier transform 3) de Hoog's algorithm (invlap), …

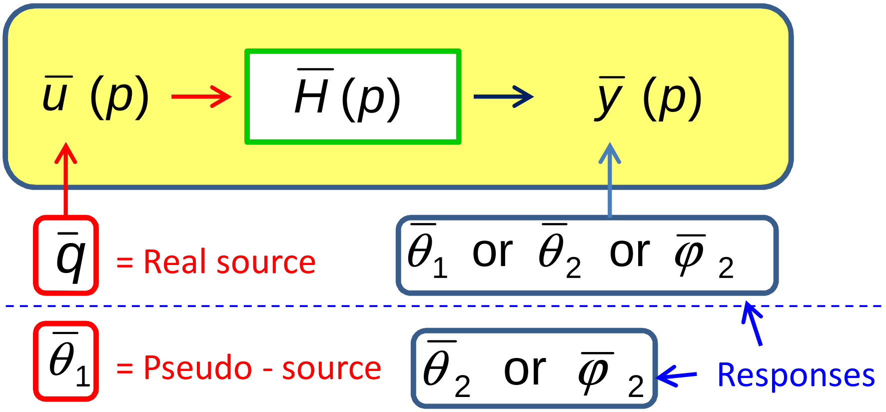


Other technique: use of the convolution product form

$$\bar\theta_1 = \bar Z_1\ \bar q \quad ; \quad \bar\theta_2 = \bar Z_2\ \bar q \quad ; \quad \bar\varphi_2 = \bar W_{flux}\ \bar q$$

impedances

Transmittance (rear face flux)



$\bar u\ (p) \rightarrow \boxed{\bar H\ (p)} \rightarrow \bar y\ (p)$

$\boxed{\bar q}$ = Real source

$\boxed{\bar\theta_1 \text{ or } \bar\theta_2 \text{ or } \bar\varphi_2}$

$\boxed{\bar\theta_1}$ = Pseudo - source

$\boxed{\bar\theta_2 \text{ or } \bar\varphi_2}$

Responses

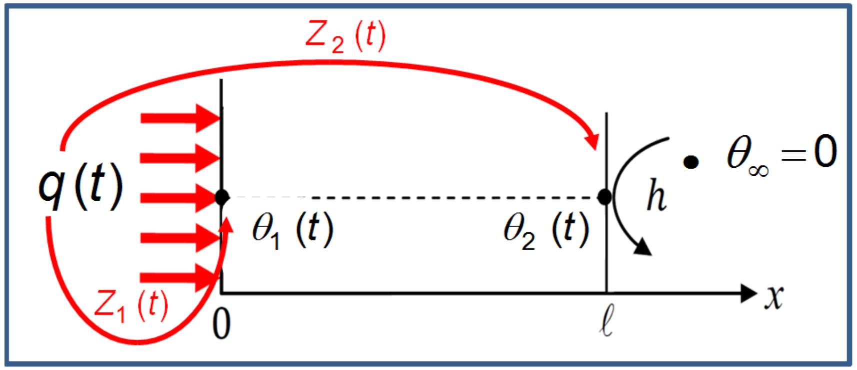
$$\bar\theta_2 = \frac{\bar Z_2}{\bar Z_1}\ \bar\theta_1 = \bar W\ \bar\theta_1 \qquad \bar\varphi_2 = \frac{\bar W_{flux}}{\bar Z_1}\ \bar\theta_1 = \bar Y\ \bar\theta_1$$

Transmittance (temperature)

Admittance

## Specific case (impedance):

$$u \equiv q \;;\; y \equiv \theta \;\; (\theta_1 \text{ or } \theta_2) \;\;;\; H \equiv Z \; (Z_1 \text{ or } Z_2)$$



**Property 2:**

$$\bar{\theta} = \bar{Z}\;\bar{q} \quad \Leftrightarrow \quad \theta\,(t) = \int_0^t Z\,(t')\;q\,(t-t')\;\mathrm{d}t'$$

**simple product** **convolution product**

**Question:** Can we write conversely $\bar{q} = \bar{Y}\;\bar{\theta}$ with $\bar{Y} = 1/\bar{Z}$ ?

$$q\,(t) = \int_0^t Y\,(t')\;\theta\,(t-t')\;\mathrm{d}t'$$

**2nd principle of thermodynamics:** $Z\,(t) \geq 0$ (impulse response)

**Property 3 :**

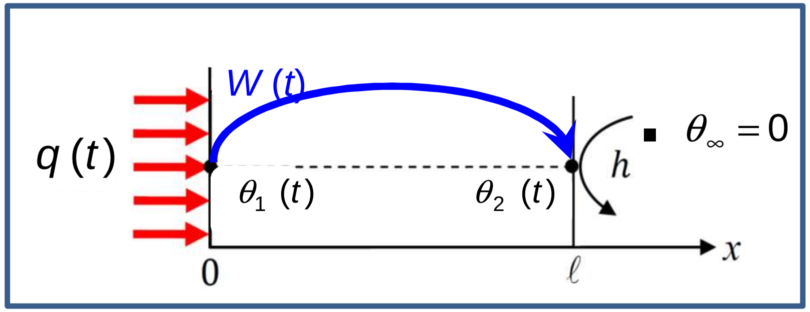
$$\frac{\mathrm{d}\bar{Z}}{\mathrm{d}p} = \frac{\mathrm{d}}{\mathrm{d}p}\left(\int_0^\infty \exp\,(-\,p\;t)\;y\;(t)\;\;\mathrm{d}t\right) = -\,p\,\bar{Z}\,(p) \leq 0 \;\;\Rightarrow \frac{\mathrm{d}\bar{Y}}{\mathrm{d}p} = \frac{\mathrm{d}(1/\bar{Z})}{\mathrm{d}p} \geq 0$$

**Result:** a **thermal admittance** $Y\;(t)$ (with respect to a temperature response) **does not exist**
It is the solution of an **inverse (ill-posed) problem**

**Causality property**:

**Heat power** source (cause) **before** any **temperature** variation (consequence) in the system

## 4. Practical calculation of a convolution product and experimental deconvolution



Specific case (transmittance):

$$H \equiv W\ ,\ u \equiv \theta_1,\ \ y \equiv \theta_2$$

**response** **transmittance** **unique pseudo source**

$$\theta_2(\,t\,) \;=\; W\;(\,t\,) * \;\theta_1\,(t)$$

$$= \int_0^t \theta_1(t - t')\,W\,(t')\,\mathrm{d}t'$$

$$z_i = \frac{1}{\Delta t}\int_{t_{i-1}}^{t_i} z\,(t)\,\mathrm{d}t \approx \frac{1}{2}\left(z\,(t_{i-1}) + z\,(t_i)\right)$$

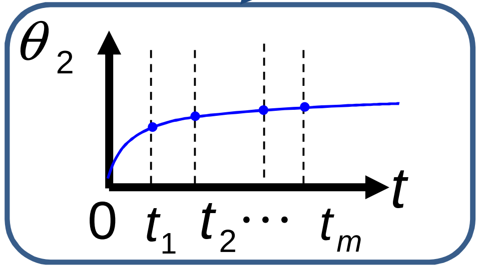
for $z\,(t) = \theta_1$ or $W$

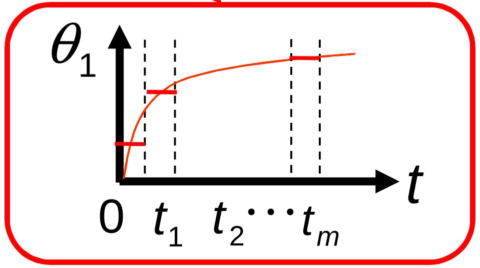
$$\theta_2(\,t_i\,) \approx \Delta t \sum_{j\,=\,1}^{m} \theta_{1,\,i-j+1}\; W_j$$

$$t_0 = 0\,;\, t_i = i\ \Delta t \quad \text{for} \quad i = 1 \text{ to } m$$

$$\Delta t = t_{final}\,/\,m$$

**sampled** **averaged over 1 time interval**





# Vector/matrix form of a convolution product

Continuous time domain:

$$\boxed{y\,(t)\; = H\,(t) * u\,(t)}$$

$$\boldsymbol{y} \;=\; \begin{pmatrix} y\,(t_1) \\ y\,(t_2) \\ \vdots \\ y\,(t_m) \end{pmatrix} \qquad \boldsymbol{H} \;=\; \begin{pmatrix} H_1 \\ H_2 \\ \vdots \\ H_m \end{pmatrix} \quad \text{and} \quad \boldsymbol{u} \;=\; \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{pmatrix}$$

**instant values**

**time averaged values over each time interval**

Introduction of a matrix function **M** (.) that depends on a time function $z\,(t)$ and on its parameterization time step $\Delta t$ :

$$\mathbf{M}\;(\boldsymbol{z}) \;\equiv\; \Delta t \begin{bmatrix} z_1 & & & & \\ z_2 & z_1 & & 0 & \\ z_3 & z_2 & z_1 & & \\ \vdots & \vdots & \vdots & \ddots & \\ z_m & z_{m-1} & z_{m-2} & \cdots & z_1 \end{bmatrix}$$

$$z_i \;=\; \frac{1}{\Delta t}\int_{t_{i-1}}^{t_i} z\,(t)\,\mathrm{d}t \;\approx\; \frac{1}{2}\big(z\,(t_{i-1}) + z\,(t_i)\big)$$

In **maths**, **M** ($\boldsymbol{z}$) is a **lower triangular Toeplitz matrix**

In heat transfer (**physics**) : $z\,(t) = H\,(t)$ or $u\,(t)$

Impulse response $H\,(t) \geq 0$ → coefficients of **M** ($\boldsymbol{H}$) are **non-negative**

Vector/matrix form of convolution product :

$$\boxed{\boldsymbol{y} = \mathbf{M}\,(\boldsymbol{u})\;\boldsymbol{H} = \mathbf{M}\,(\boldsymbol{H})\;\boldsymbol{u}}$$

Specific case (transmittance):

$$H \equiv W \;,\;\; u \;\equiv \theta_1, \;\;\; y \;\equiv \theta_2$$

$$\boldsymbol{\theta}_2 = \mathbf{M}\,(\boldsymbol{\theta}_1)\;\boldsymbol{W} = \mathbf{M}\,(\boldsymbol{W})\;\boldsymbol{\theta}_1$$

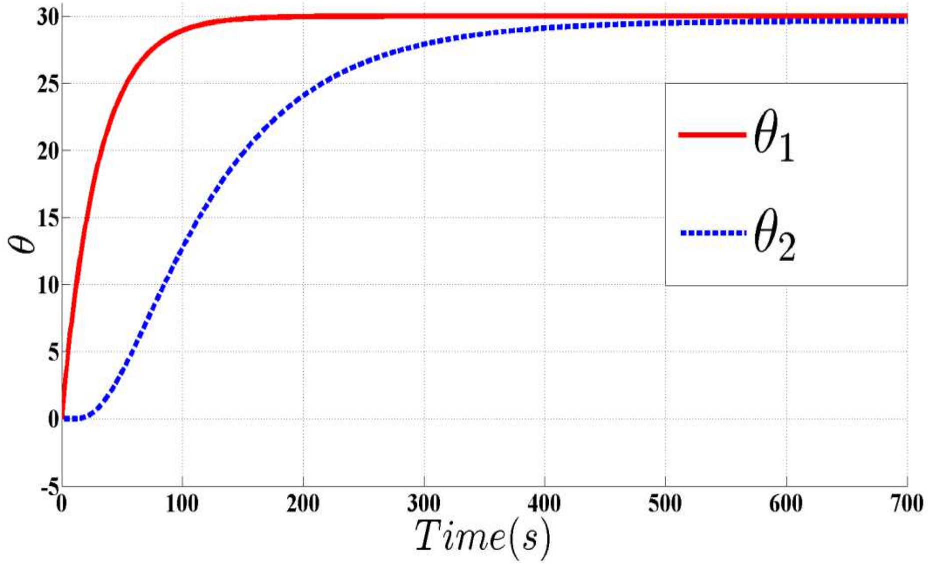
# Validation: Numerical Inversion of Laplace Transforms by de Hoog's algorithm (Invlap)

$$\theta_1(t) = \left(1 - e^{-\frac{t}{\tau}}\right)\theta_1^{ss} \quad \text{with} \quad \tau = 30s \quad ; \quad \theta_1^{ss} = 30\,°C \quad \text{and} \quad \Delta t = 0.5s$$

| $t_f$ (s) | $\ell$ (mm) | $h$ (W.m$^{-2}$.K$^{-1}$) | $\lambda$ (W.m$^{-1}$.K$^{-1}$) | $\rho c$ (kJ.m$^{-3}$.K$^{-1}$) |
|---|---|---|---|---|
| 700 | 50 | 10 | 43 | 3666 |

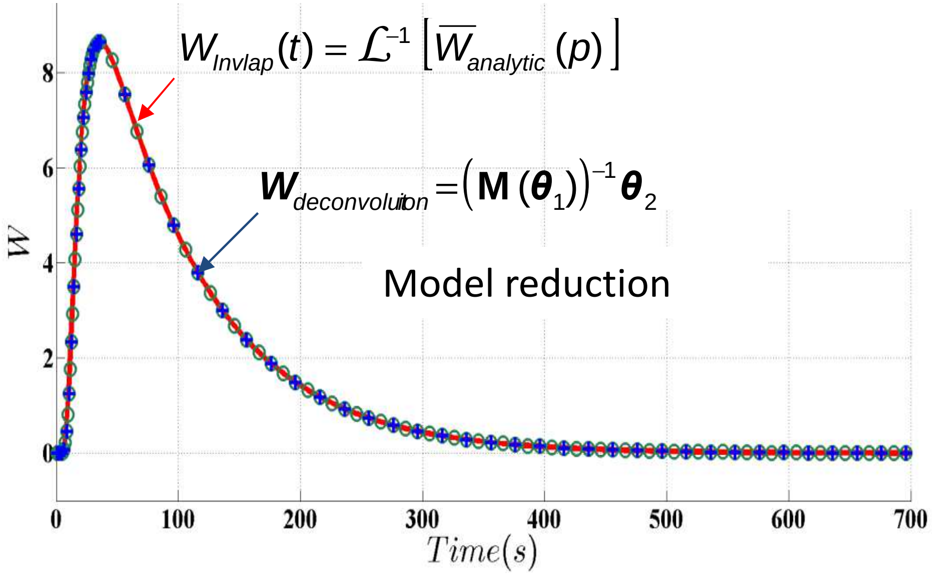
## Comparison: analytical W and identified W from synthetic profiles (COMSOL)

### Comparison without noise:



Temperature profiles (COMSOL)

**Cross - validation**



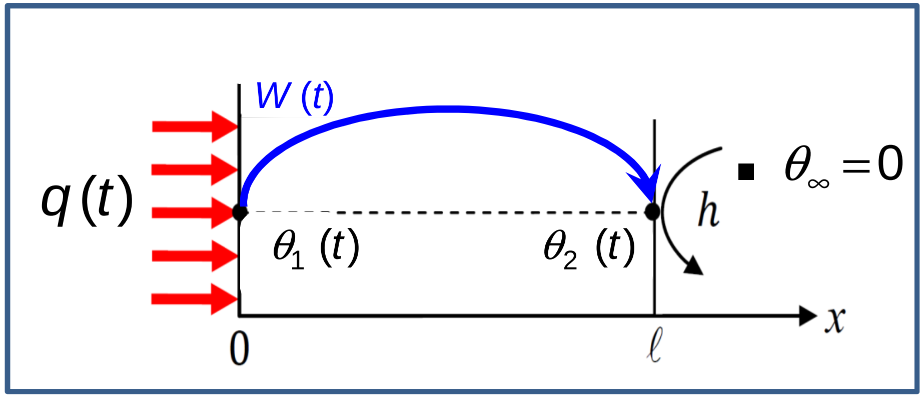
$$W_{Invlap}(t) = \mathcal{L}^{-1}\left[\overline{W}_{analytic}(p)\right]$$

$$\boldsymbol{W}_{deconvolution} = \left(\mathbf{M}\,(\boldsymbol{\theta}_1)\right)^{-1}\boldsymbol{\theta}_2$$

Model reduction

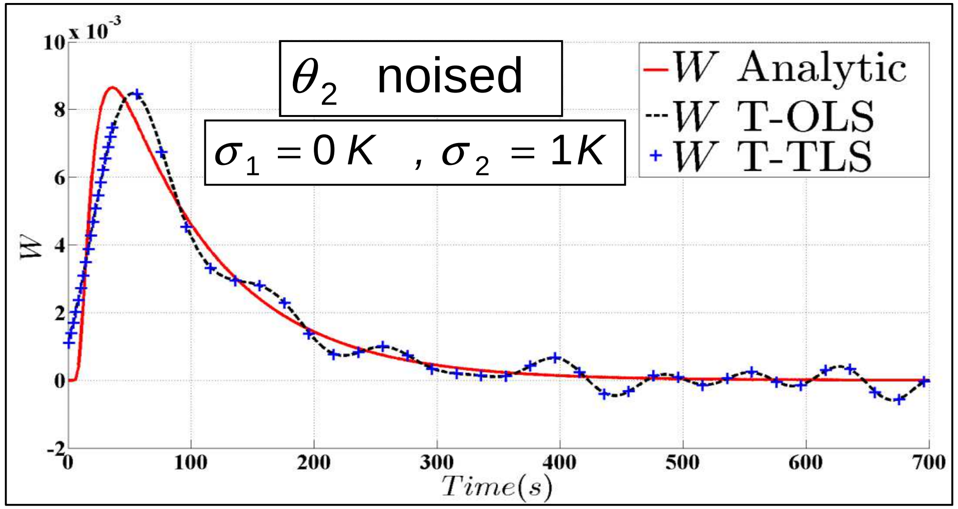
Analytical and identified W

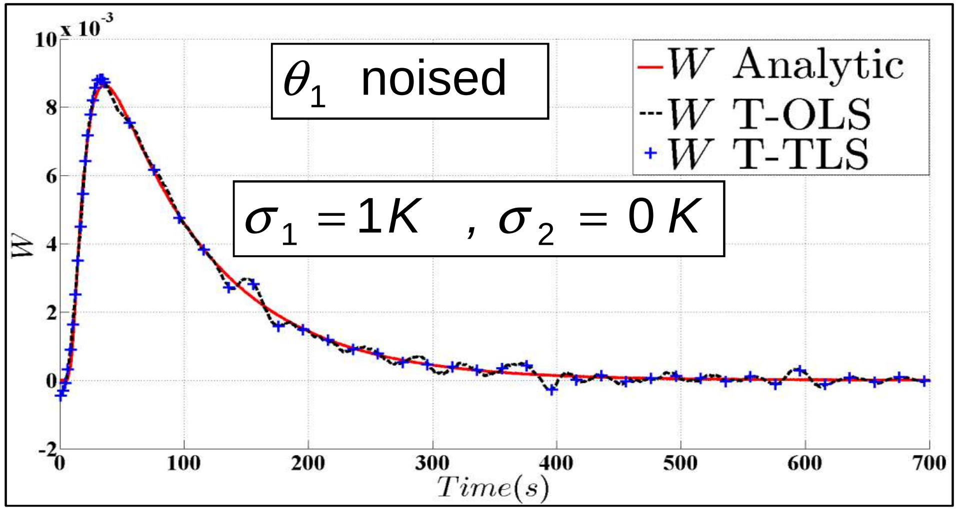
# Effect of noise on identified transmittance (simulations, 1D configuration)

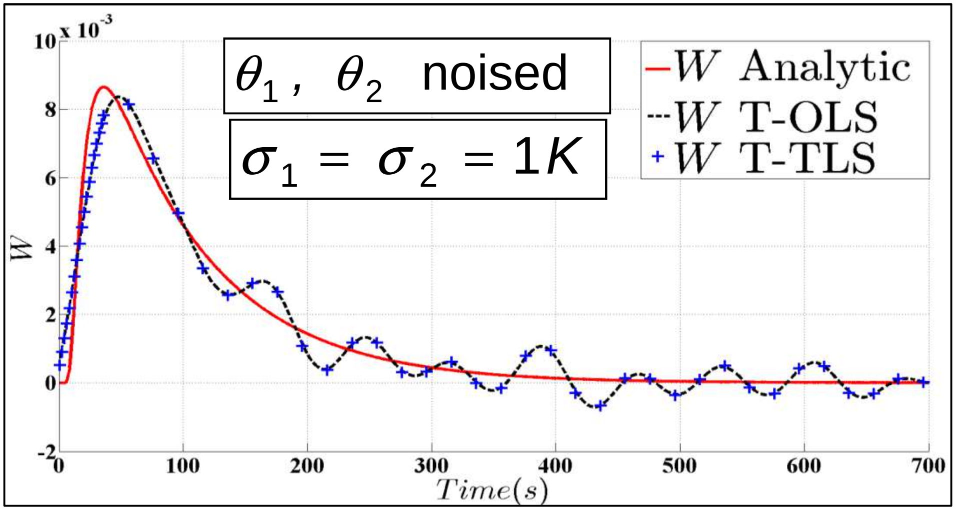


$$\hat{W} = \left(\mathbf{M}(\boldsymbol{\theta}_1^{exp})\right)^{-1} \boldsymbol{\theta}_2^{exp} \Rightarrow \text{explosion}$$

**Regularization**: TSVD version of $\left(\mathbf{M}(\boldsymbol{\theta}_1^{exp})\right)^{-1}$





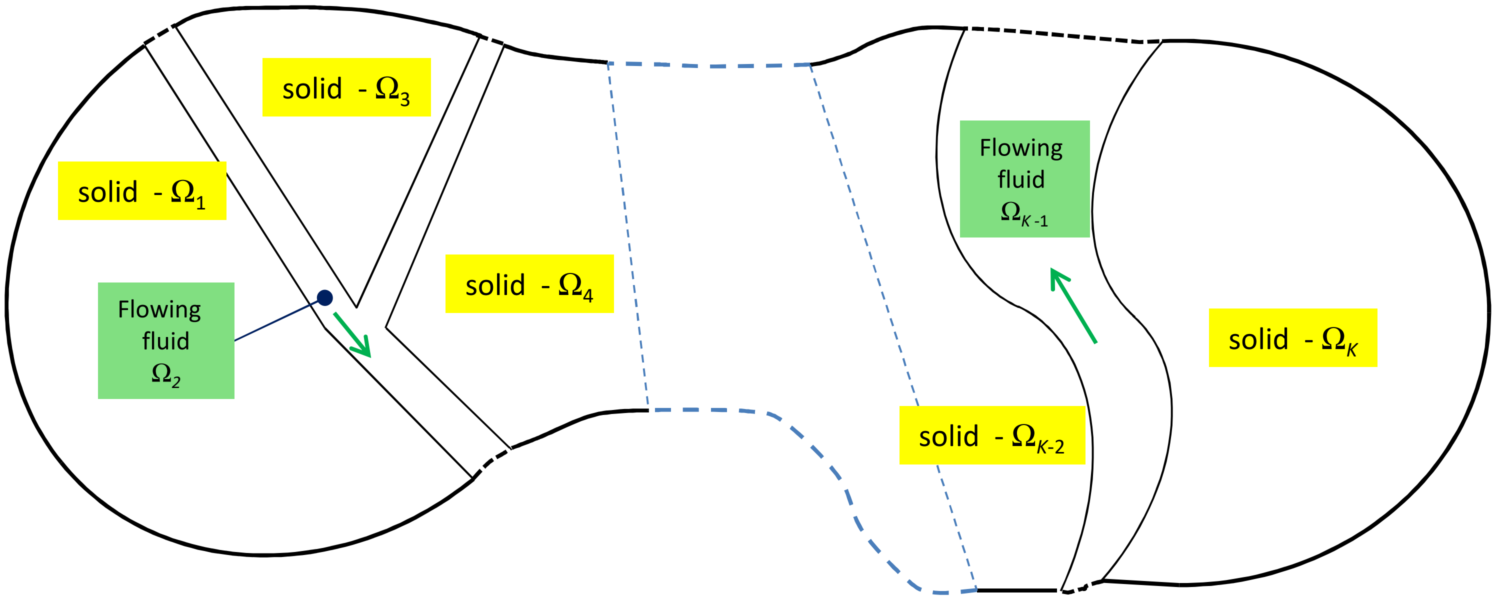


- Noise on the response θ2 more penalizing than noise on the source θ1.

## 5. Laplace transform and 3 D heat transfer

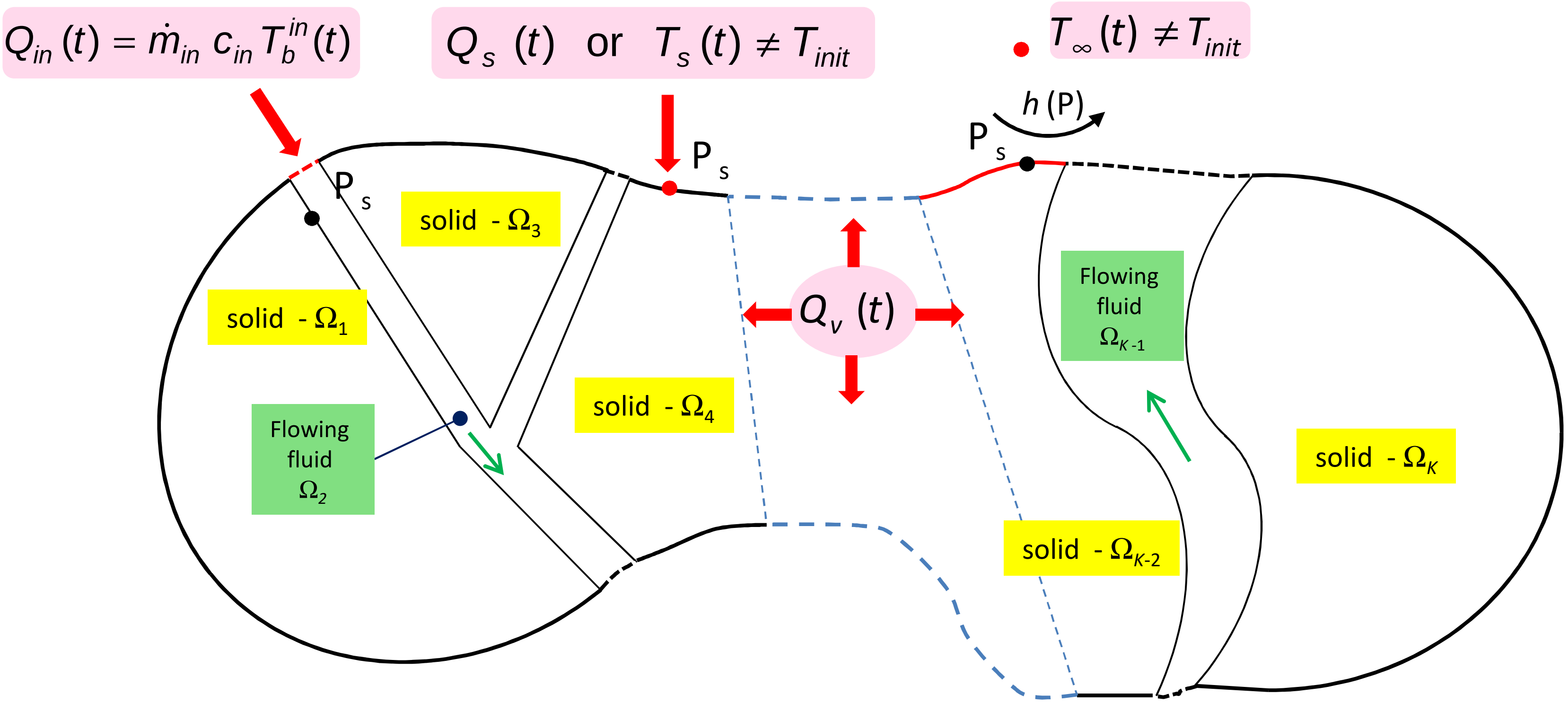
What happens in 3 D transient heat transfer ?

Material multicomponent system = $K$ solid or fluid domains



Assumptions: **time constant thermophysical properties** and **velocity field**

## Initial uniform temperature field or steady state conditions + one single separable unsteady thermal excitation



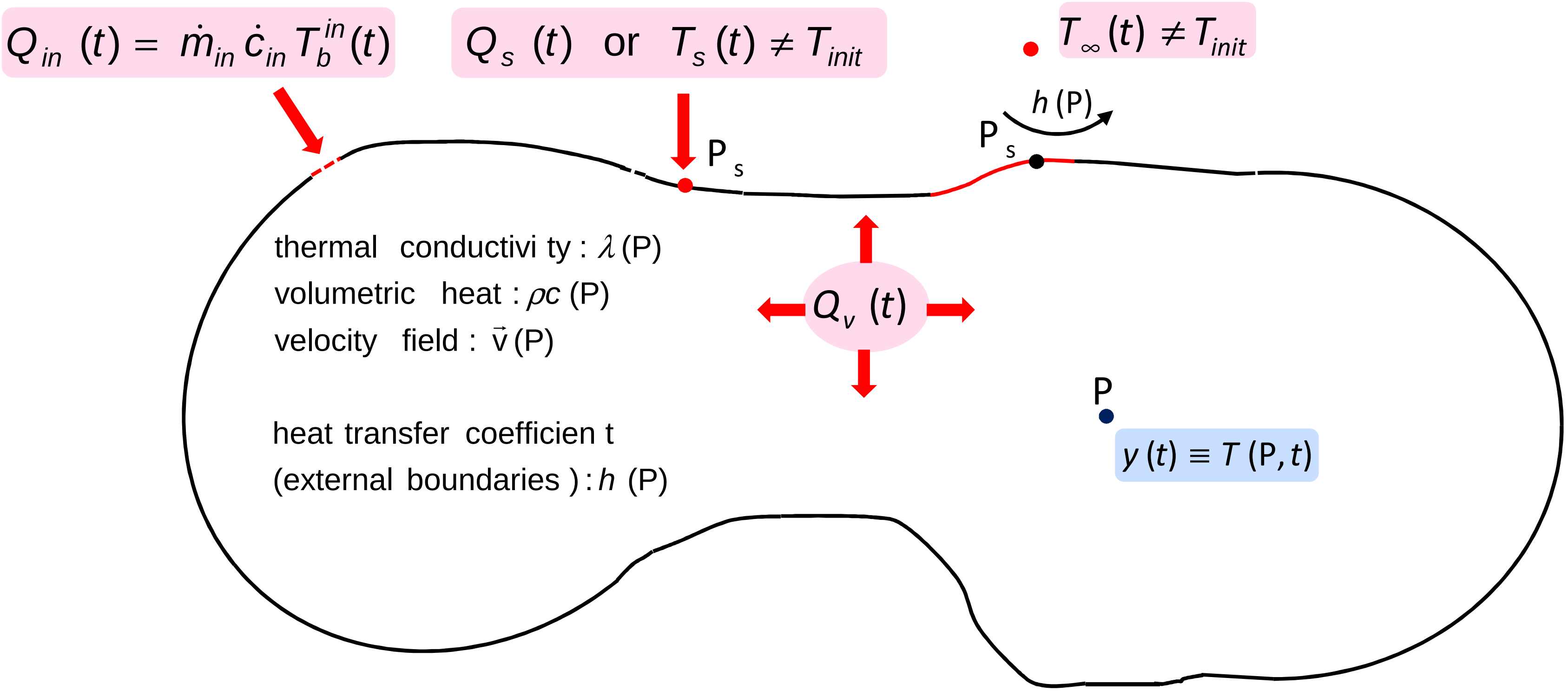
**Time part of thermal excitation** $u\,(t)$ (starts at time t = 0) :

- volumetretric heat source $Q_v\ (t)$
- surface heat or temperature source $Q_s\ (t)$ or $T_s\,(t)$
- change of external fluid temperature $T_\infty(t) \neq T_{init}$
- change of temperature at one fluid inlet $T_b^{in}(t)$

**Fixed geometrical support**:

- point
- line
- surface
- volume

# Change of perspective: one single **heterogeneous fluid** in **one single domain** (if solid part : zero velocity)



$Q_{in}\ (t) = \dot{m}_{in}\, \dot{c}_{in}\, T_b^{in}(t)$

$Q_s\ (t)$ or $T_s\,(t) \neq T_{init}$

$T_\infty(t) \neq T_{init}$

thermal conductivi ty : $\lambda$ (P)

volumetric heat : $\rho c$ (P)

velocity field : $\vec{v}$ (P)

heat transfer coefficien t (external boundaries ) : $h$ (P)

$Q_v\ (t)$

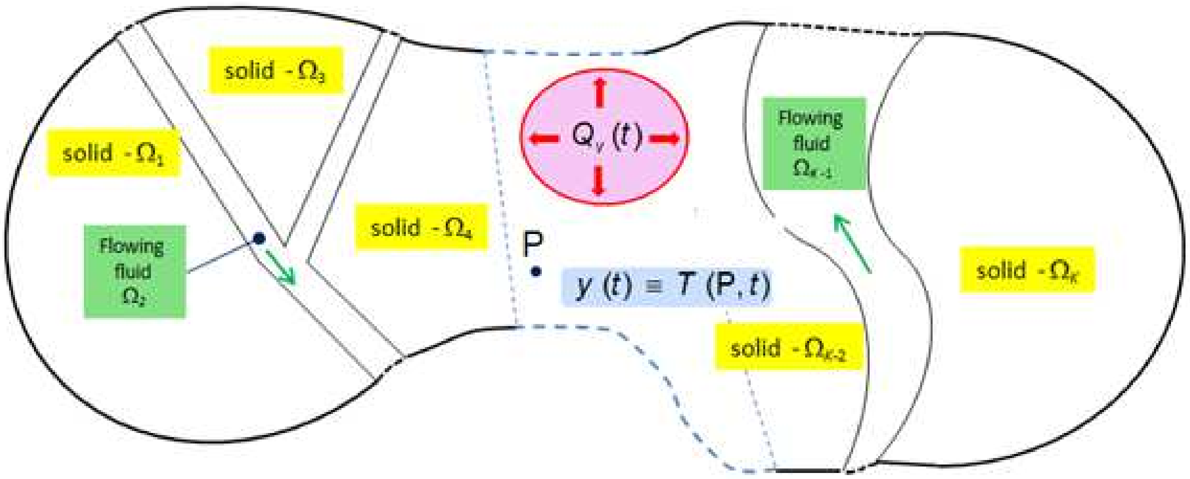
$y\,(t) \equiv T\,(\mathrm{P}, t)$

**Transient separable thermal excitation :**

**Point response at any point P :**

$$u\ (t) \quad \Rightarrow \quad y\ (t)\ \equiv\ T\ (\mathrm{P}, t)$$

# Recap



Physical system:

Set of solids AND fluid(s):

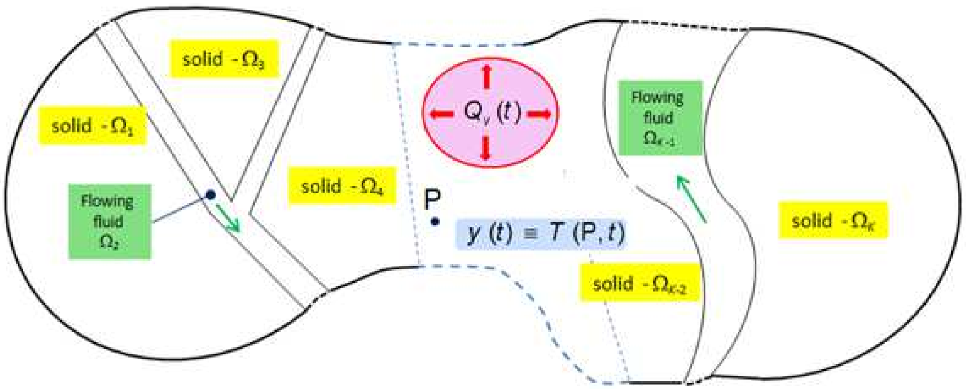
3D forced convection
with constant velocities
(in time but not in space)

P = ANY point in the system

One single thermal excitation defined by its support and separable

*Assumptions :* Transient heat equation + boundary conditions with time-invariant coefficients + uniform initial temperature or steady state (the system is Linear and also Time-Invariant LTI)

$$\underbrace{\boxed{\rho c(P)\frac{\partial T}{\partial t}(P,t)}}_{\text{Transient}} + \underbrace{\boxed{\rho c(P)\vec{u}\,(P).\,\vec{\nabla}T\,(P,t)}}_{\text{Advection}} = \underbrace{\boxed{\vec{\nabla}.\,\left(\lambda(P)\,\vec{\nabla}T\,(P,t)\right)}}_{\text{Conduction}} + \underbrace{\boxed{\frac{Q_v\;(t)}{V_{\text{source}}}f\,(P)}}_{\text{Internal source}}$$



Temperature rise at any point P:

$$\theta\,(\mathrm{P},t)=T\,(\mathrm{P},t)-T_{init}\,(\mathrm{P})$$

Its Laplace transform :

$$\bar{\theta}\,(\mathrm{P},p)=\int_0^\infty \exp\,(-p\,t)\;\theta\,(\mathrm{P},t)\;\mathrm{d}t$$

**Laplace parameter**

*Assumptions :* Transient heat equation + boundary conditions with time-invariant coefficient + uniform initial temperature (the system is Linear and also Time-Invariant LITI)

$$\underset{\text{Transient}}{\boxed{\rho c(\mathrm{P})\frac{\partial T}{\partial t}(\mathrm{P},t)}}+\underset{\text{Advection}}{\boxed{\rho c(\mathrm{P})\vec{u}\,(\mathrm{P}).\,\vec{\nabla}T\,(\mathrm{P},t)}}=\underset{\text{Conduction}}{\boxed{\vec{\nabla}.\left(\lambda(\mathrm{P})\,\vec{\nabla}T\,(\mathrm{P},t)\right)}}+\underset{\text{Internal source}}{\boxed{\frac{Q_v\,(t)}{V_{\text{source}}}f\,(\mathrm{P})}}$$

*Consequences :*Laplace transformed heat equation[4] (no time derivative)

$$\underset{\text{Transient}}{\boxed{\rho c(\mathrm{P})\,p\,\bar{\theta}(\mathrm{P},p)}}+\underset{\text{Advection}}{\boxed{\rho c(\mathrm{P})\vec{u}\,(\mathrm{P}).\,\vec{\nabla}\bar{\theta}\,(\mathrm{P},p)}}=\underset{\text{Conduction}}{\boxed{\vec{\nabla}.\left(\lambda(\mathrm{P})\,\vec{\nabla}\bar{\theta}\,(\mathrm{P},p)\right)}}+\underset{\text{Internal source}}{\boxed{\frac{\bar{Q}_v\,(p)}{V_{\text{source}}}f\,(\mathrm{P})}}$$

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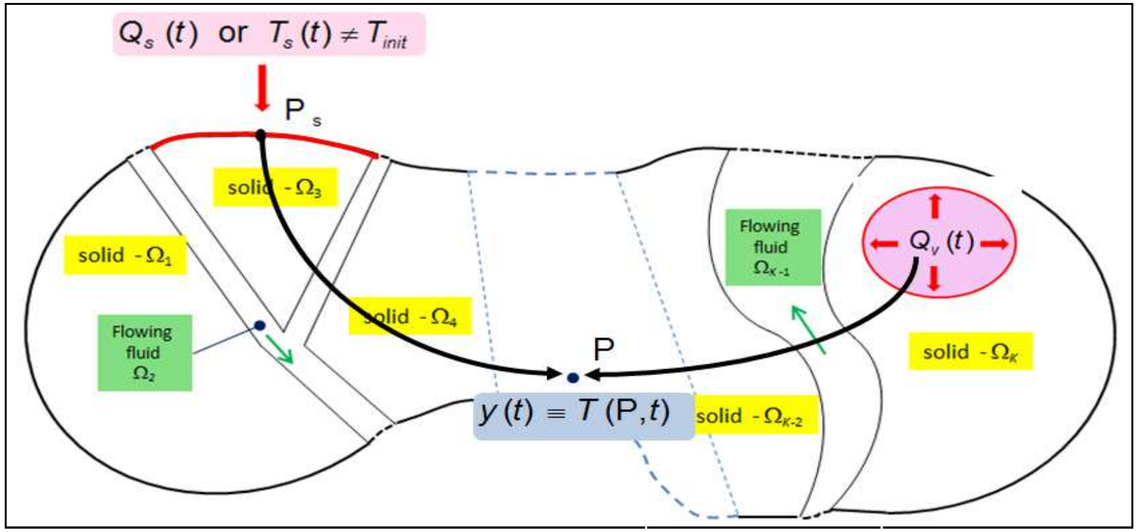
[4] W. Al Hadad, D. Maillet, Y. Jannot, Modeling unsteady diffusive and advective heat transfer for linear dynamical systems: A transfer function approach, International Journal of Heat and Mass Transfer 115 (2017) 304–313.

## *Linear system with a single excitation*

$\Rightarrow$ Temperature or flux response at any point P in the system

= simple product (Laplace domain)

$$\bar{y}\,(\mathrm{P},p) = \bar{H}\;(\mathrm{P},p)\;\;\bar{u}\;(p)$$



or convolution product (time domain)

**excitation**

$$y\,(\mathrm{P},t) = H\;(\mathrm{P},t)\,*\;\;u\;(t) = \int_0^t\;H\;(\mathrm{P},\;t-t')\;u\;(t')\;\mathrm{d}t'$$

**Forced response**

**Relative (transient) excitation** $u\,(t)$ **:**

$u\;(t) = Q_v\,(t) - Q_v^{init}$ or $Q_s\,(t) - Q_s^{init}$

or $T_s\,(t)$ - $T_{init}\,(\mathrm{P}_s)$ or $T_\infty(t)$ - $T_\infty^{init}$

or $T_b^{in}\,(t)$ - $T_b^{in,init}$

**Impulse response**

$$\xrightarrow{\quad}\;\boxed{H\;(\mathrm{P}\;,\;t)}\;\xrightarrow{\quad}$$

« init »= initial steady state or uniform temperature field

**Response** $y\;(t)$ **in any specific point P :**

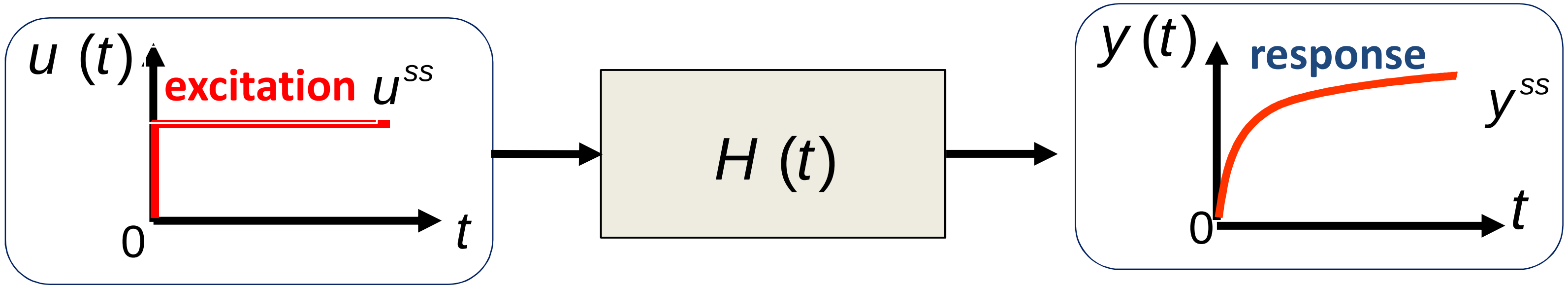
$y\;(t) = \theta\,(\mathrm{P},t) = T\;(\mathrm{P},t)$ - $T_{init}\,(\mathrm{P})$

or local heat flux in any direction $\varphi_x\,(\mathrm{P},t)$

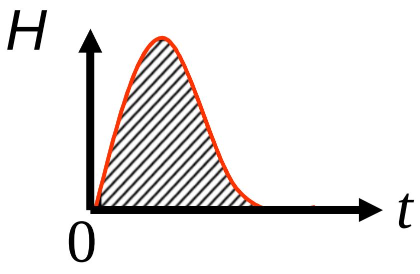
| Excitation *u* | Response *y* | Transfer function *H* |
|---|---|---|
| Power source *Q* (watts) | Temperature difference $\theta$ (kelvins) | Impedance Z (K.J$^{-1}$) |
| Temperature difference $\theta$ (kelvins) | Temperature difference $\theta$ (kelvins) | Transmittance *W* (s$^{-1}$) |
| Power source *Q* (watts) | Rate of heat flow $\Phi$ (watts) | Transmittance *W* (s$^{-1}$) |
| Temperature difference $\theta$ (kelvins) | Rate of heat flow $\Phi$ (watts) | Admittance *Y* (W.K$^{-1}$.s$^{-1}$) |

## 6. Laplace transform and steady state transfer functions

$$\boxed{y\,(\mathrm{P},t) = H\ (\mathrm{P},t) *\ u\ (t) = \int_0^t\ H\ (\mathrm{P},\ t-t')\,u\ (t')\ \mathrm{d}t'}$$



**Steady state version** (ss)
of a transfer function



$$\boxed{H^{ss} = \int_0^{\infty} H\,(t)\ dt}$$

time
distribution

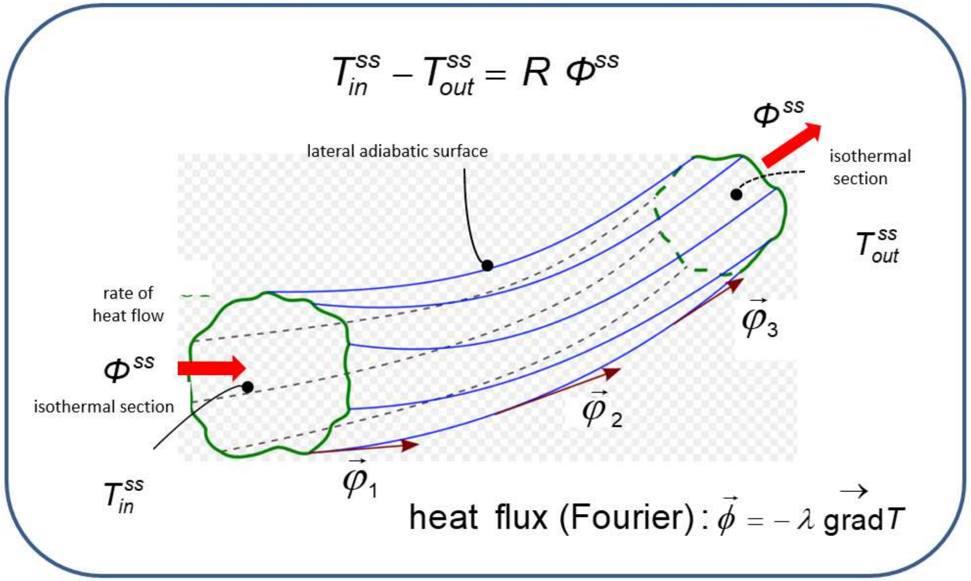
$$\boxed{y^{ss} = H^{ss}\,u^{ss}}$$

asymptotic
values

## Traditional definition of a thermal resistance in steady state regime

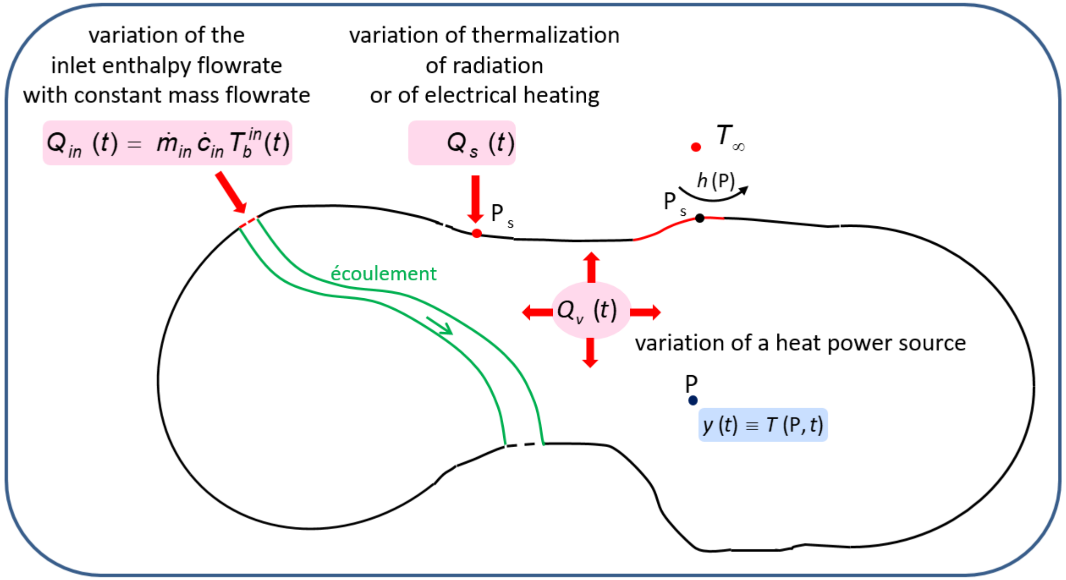
**Assumption** :
A flux pipe exists between
2 isothermal surfaces

$\Phi^{ss}$ : steady state rate of heat flow



$$T_{in}^{ss} - T_{out}^{ss} = R\ \Phi^{ss}$$

heat flux (Fourier) : $\vec{\phi} = -\lambda\ \overrightarrow{\text{grad}}\,T$



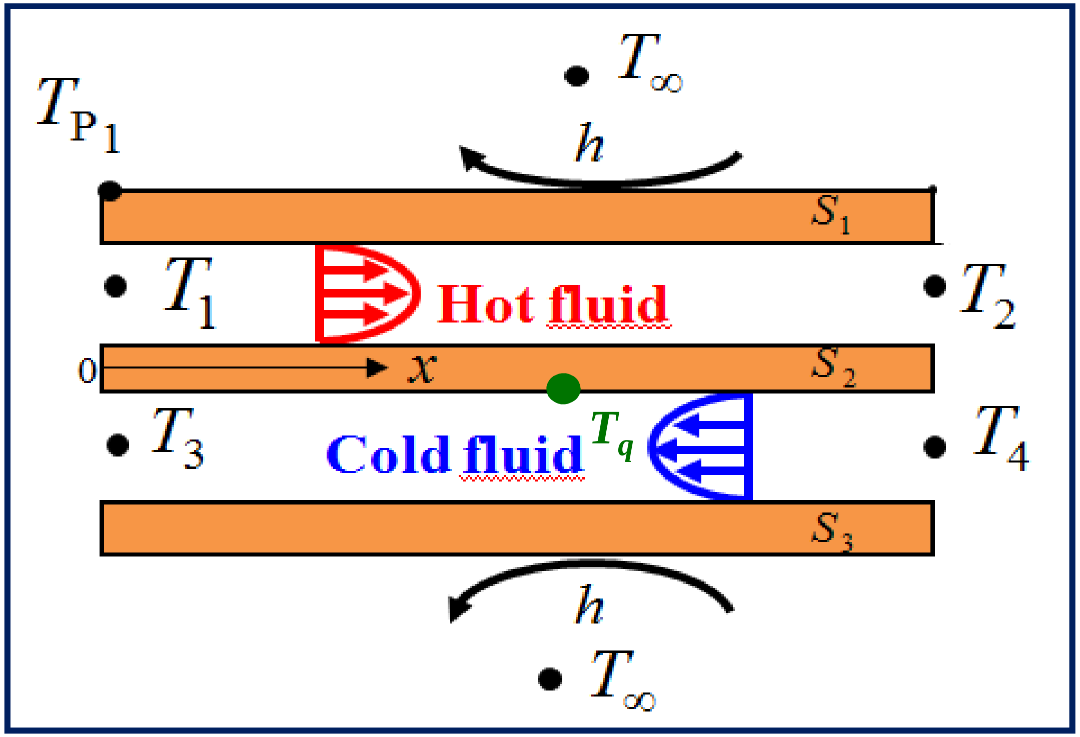
## Generalized resistance : no flux pipe , no isothermal surface

$Q^{ss}$ : source (= cause)

$$\boxed{T_2^{SS} - T_1^{SS} = Z^{SS}\ (\underbrace{Q_2^{SS} - Q_1^{SS}})}$$

variation of thermal power (watts)
(**thermodynamical conversion**
from a non thermal energy)
between 2 steady states

# 7. Applications to conjugated heat transfer in heat exchangers

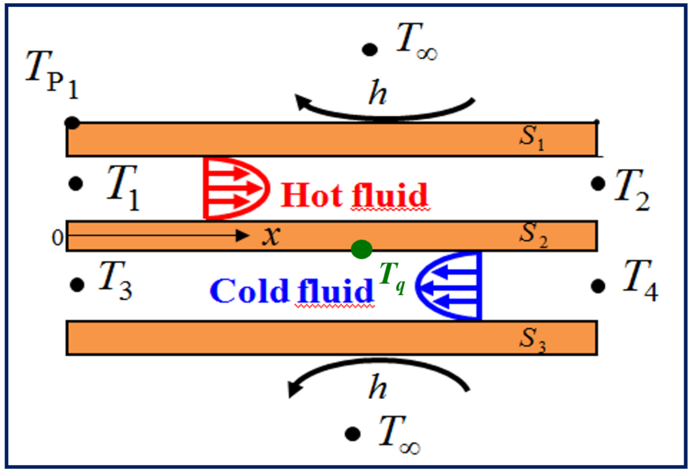


❑ Thermal regime caused by an unsteady thermal excitation somewhere and observed temperature response in any point $q$ :

$\boxed{\theta_q(t) = T_q(t) - T_{init}}$

Consequences :

$\boxed{\theta_q(t \le 0) = 0 \quad \text{and} \quad \theta_q(t > 0) \neq 0}$



# Example: perturbation of inlet temperature $T_1$ of hot fluid

$$\boxed{\mathbf{M}\ (\boldsymbol{z})\ \equiv\ \Delta t\ \begin{bmatrix} z_1 & & & & \\ z_2 & z_1 & & 0 & \\ z_3 & z_2 & z_1 & & \\ \vdots & \vdots & \vdots & \ddots & \\ z_m & z_{m-1} & z_{m-2} & \cdots & z_1 \end{bmatrix}}$$

**response** **transmittance** **unique pseudo source**

$$\boldsymbol{\theta}_q\ =\begin{pmatrix}\theta_q(t_1)\\ \theta_q(t_2)\\ \vdots\\ \theta_q(t_m)\end{pmatrix},q\ =\ 2,\ 3,\ 4\ \text{ or }\ \mathrm{P}_1$$

**instantaneous values**

$$\boxed{\boldsymbol{\theta}_q=\mathbf{M}\,(\boldsymbol{\theta}_1)\ \boldsymbol{W}_1^{\,q}\ =\mathbf{M}\,(\boldsymbol{W}_1^{\,q})\ \boldsymbol{\theta}_1}$$

$$\boldsymbol{\theta}=\begin{pmatrix}\theta_{1,1}\\ \theta_{1,2}\\ \vdots\\ \theta_{1,m}\end{pmatrix}\quad \boldsymbol{w}=\begin{pmatrix}w_1\\ w_2\\ \vdots\\ w_{1m}\end{pmatrix}$$

**averaged values**

- **First experiment:**

  - (inverse) calibration problem

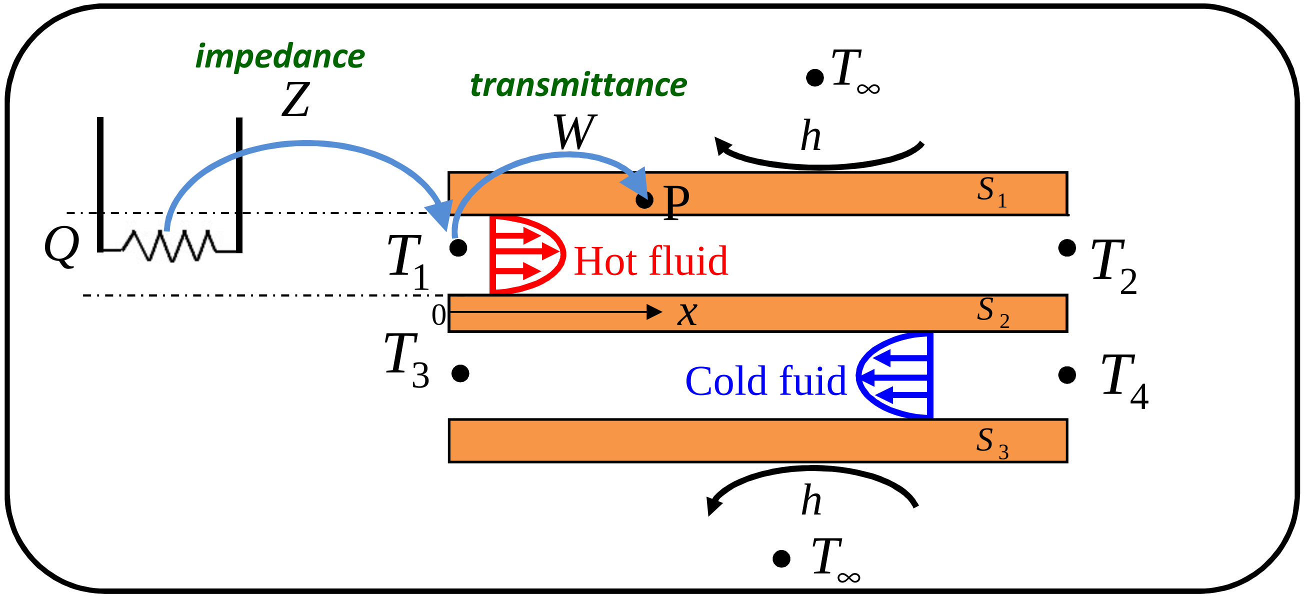
$$\underset{\text{measured}}{\boldsymbol{\theta}_q}\longrightarrow\boxed{\underset{\text{mesured}}{(\mathbf{M}\ (\boldsymbol{\theta}_1))^{-1}}}\longrightarrow\underset{\text{estimated}}{\hat{\boldsymbol{W}}_1^{\,q}}$$

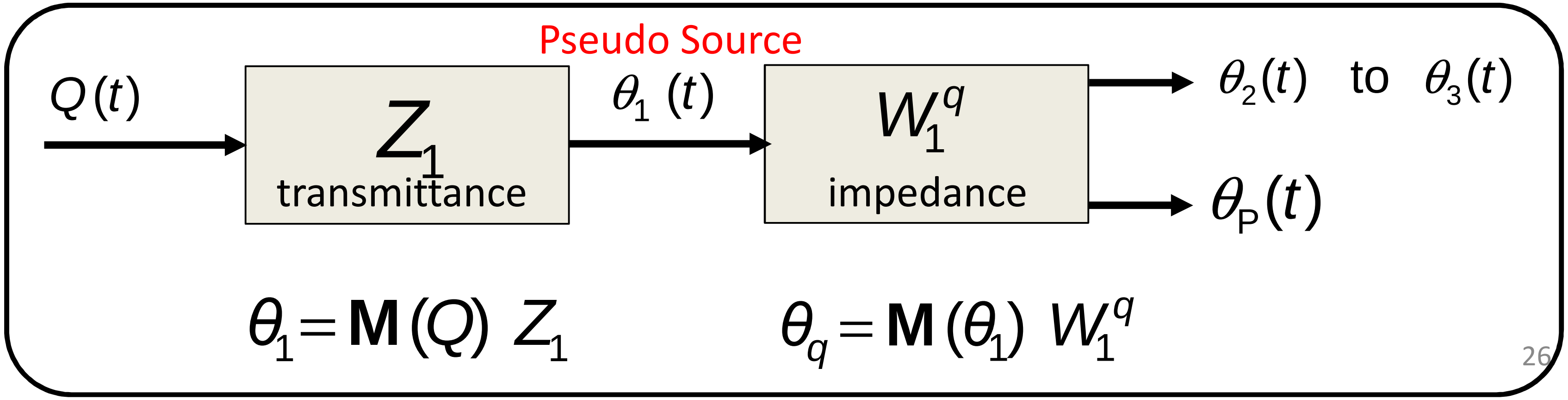
- **Following experiments:**

  - inverse use = virtual temperature sensor (= inverse PB of source estimation

$$\underset{\text{mesured}}{\boldsymbol{\theta}_q}\longrightarrow\boxed{\underset{\text{Previously estimated}}{(\mathbf{M}\ (\hat{\boldsymbol{W}}_1^{\,q}))^{-1}}}\longrightarrow\underset{\text{estimed}}{\hat{\boldsymbol{\theta}}_1}$$

# How to change the inlet temperature of one fluid in a heat exchanger without changing the flowrates ?





Pseudo Source

$Q(t)$

$Z_1$ transmittance

$\theta_1\ (t)$

$W_1^q$ impedance

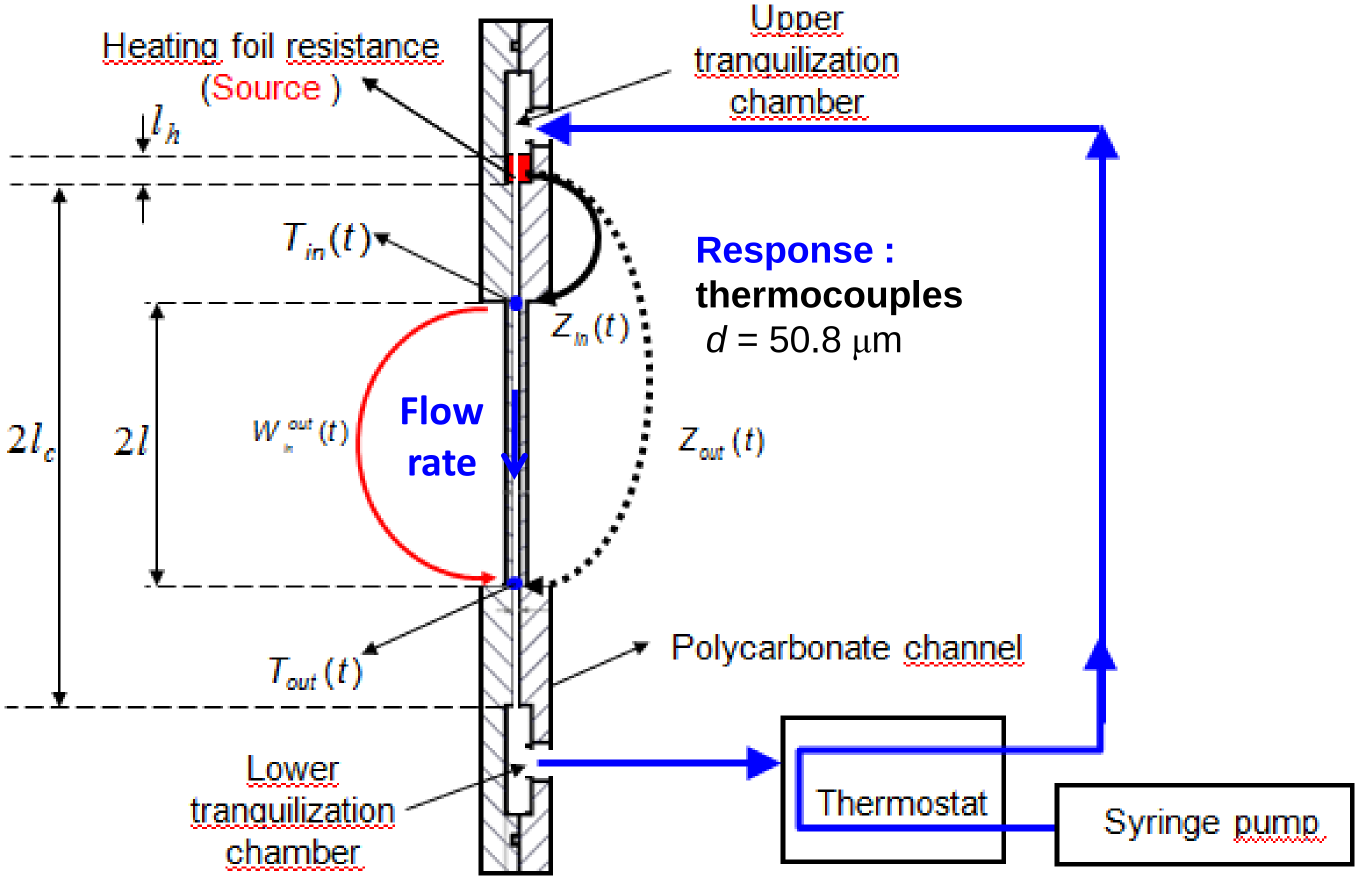
$\theta_2(t)$ to $\theta_3(t)$

$\theta_P(t)$

$$\theta_1 = \mathbf{M}(Q)\ Z_1 \qquad \theta_q = \mathbf{M}(\theta_1)\ W_1^q$$

# Example 1 : Experimental impedance/transmittance estimation for a heated flow in a thick wall channel[5] (calibration)

| $e_1 = e_2(mm)$ | $e_f(mm)$ | $2l(mm)$ | $2l_c(mm)$ | $w(mm)$ | $l_h(mm)$ |
|---|---|---|---|---|---|
| 2 | 1 | 65 | 120 | 50 | 10 |



**Response :**
**thermocouples**
$d$ = 50.8 µm

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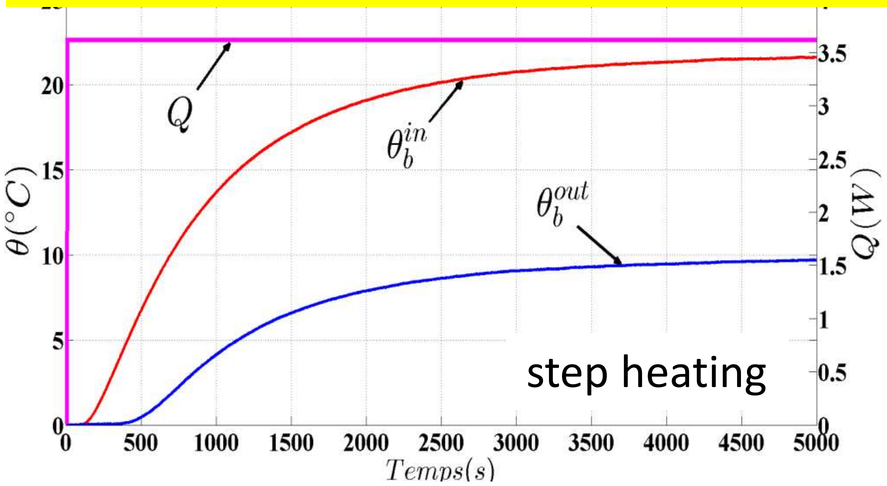
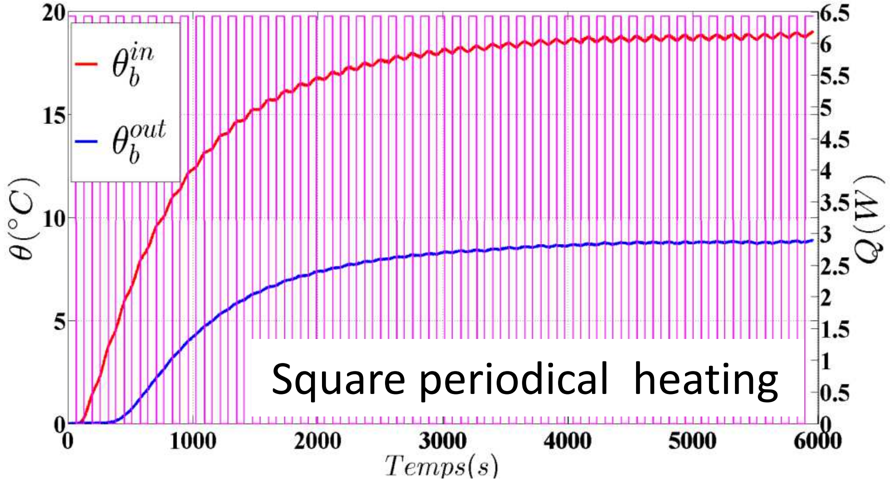
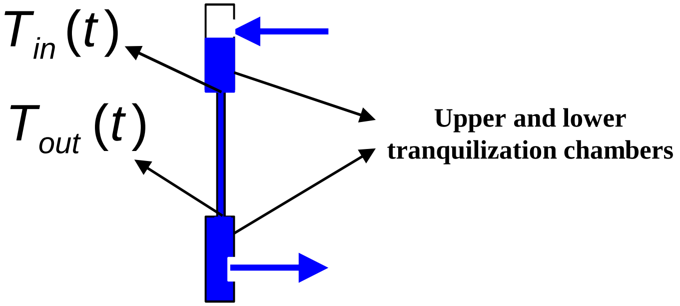
[5] W. Al Hadad, D. Maillet, Y. Jannot, Experimental transfer functions identification: Thermal impedance and transmittance in a channel heated by an upstream unsteady volumetric heat source, International Journal of Heat and Mass Transfer 116 (2018) 931–939.

## Identification of transfer function using experimental temperature recording:

**Comparison of identified $Z$ : step or periodical heating**



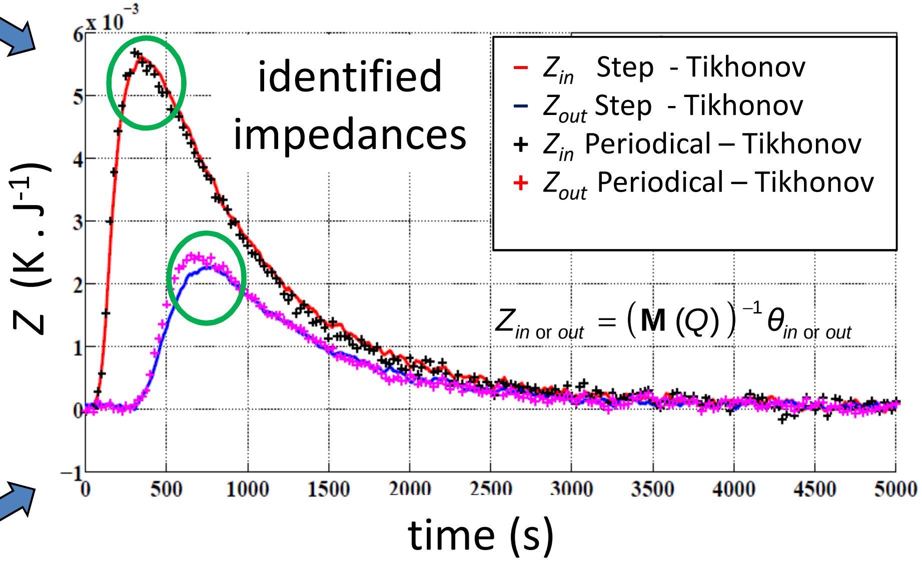




$T_{in}(t)$

$T_{out}(t)$

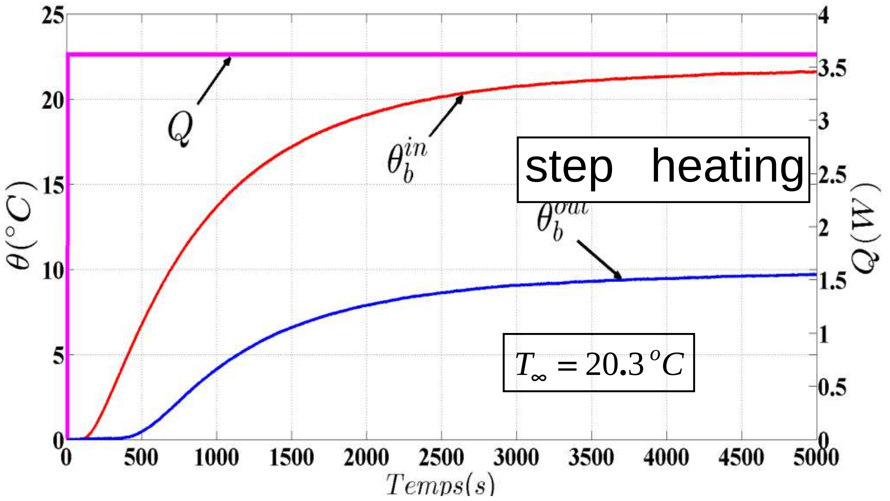
**Upper and lower tranquilization chambers**

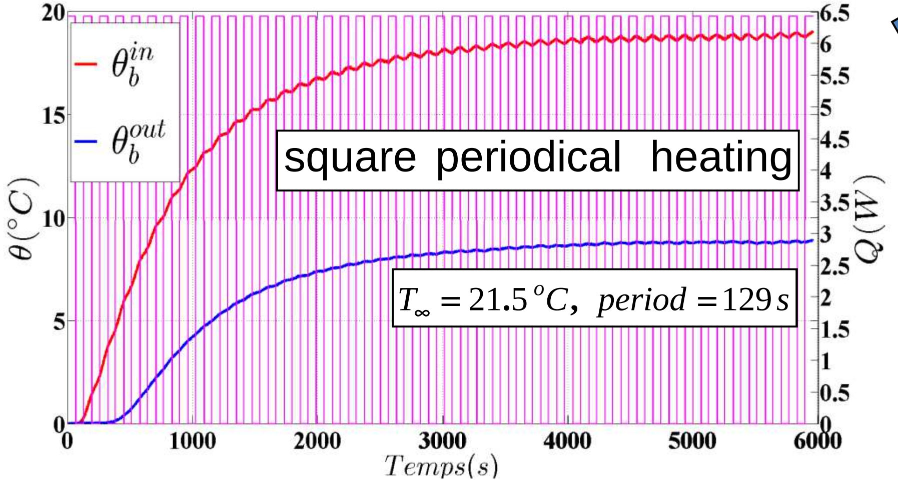


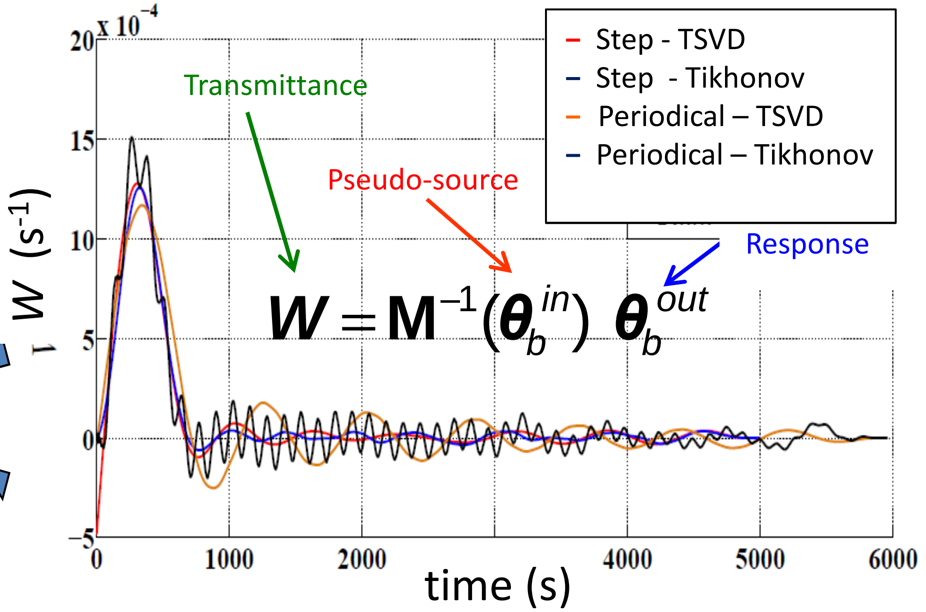
– $Z_{in}$ Step - Tikhonov
– $Z_{out}$ Step - Tikhonov
+ $Z_{in}$ Periodical – Tikhonov
+ $Z_{out}$ Periodical – Tikhonov

$$Z_{in\ or\ out} = \left(\mathbf{M}\,(Q)\right)^{-1}\theta_{in\ or\ out}$$

# Comparison of identified transmittance *W* (outlet/inlet): step or periodical heating





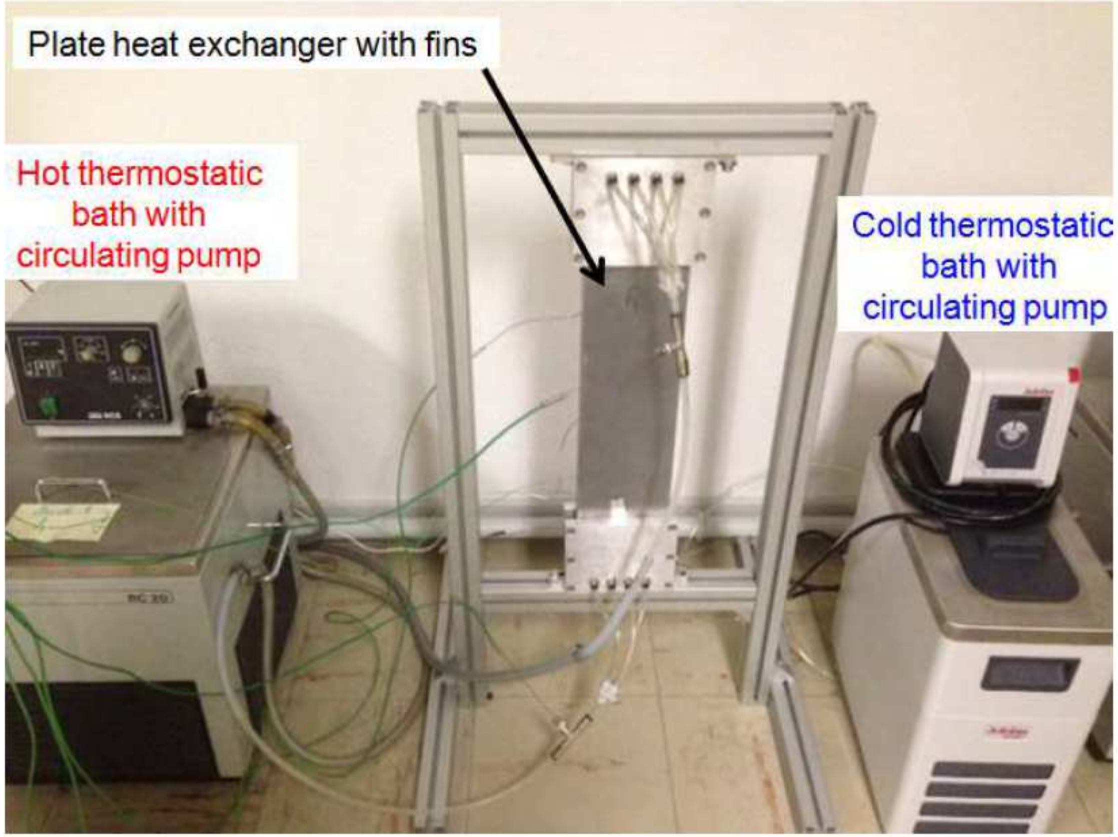


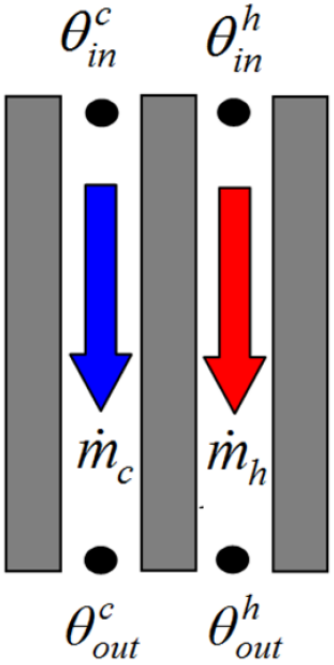
$$W = \mathbf{M}^{-1}(\boldsymbol{\theta}_b^{in})\ \boldsymbol{\theta}_b^{out}$$

**Oscillations past first peak and for long times, zero initial level hard to recover** :

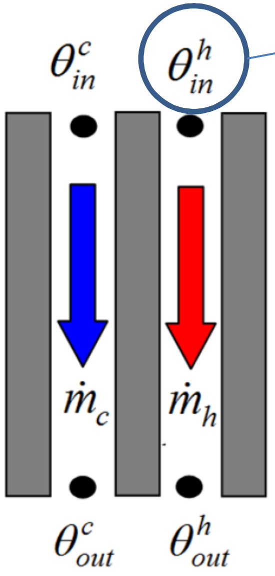
Estimation of transmittance *W* (noisy output and input) more difficult here than *Z* (noisy output and input, and bias: not completely LTI)

# Example 2: Experimental identification the model of a plate fin heat exchanger



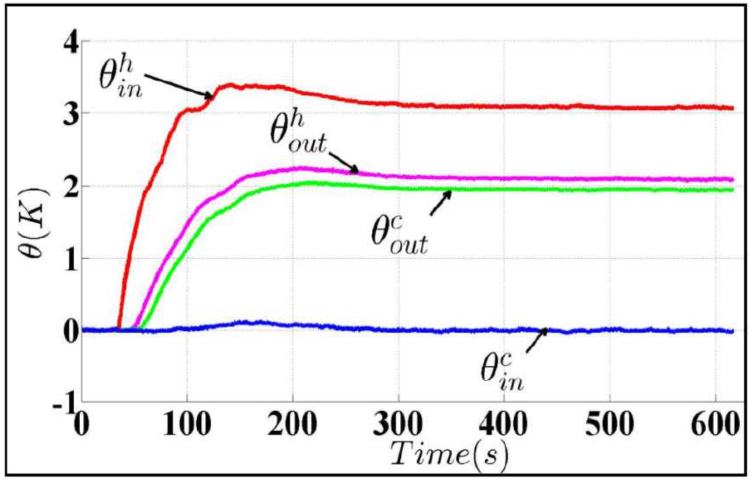


Fluids =water

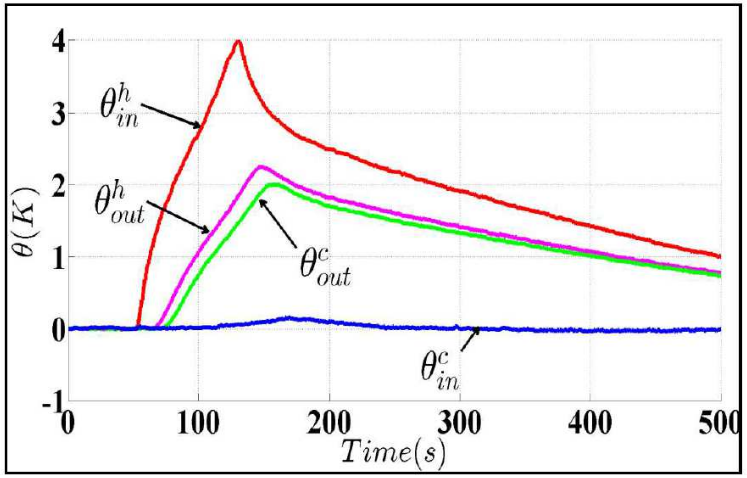


Manual control through changing setpoint temperature

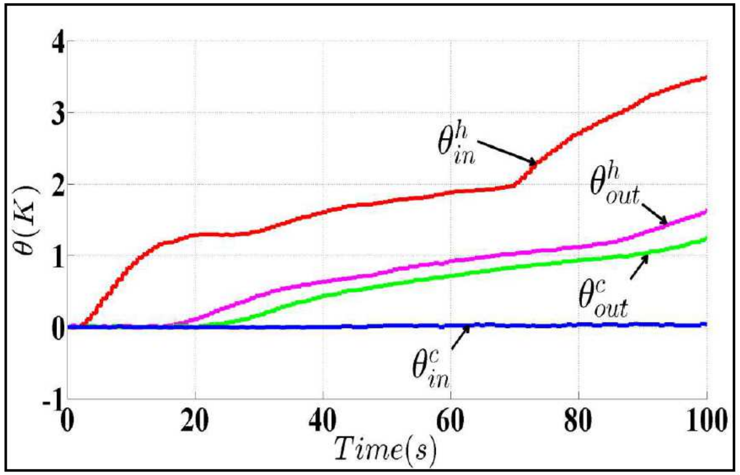
Thermostats with circulation Pump (unchanged flowrates)



Inlet/outlet thermograms – Experiment 1

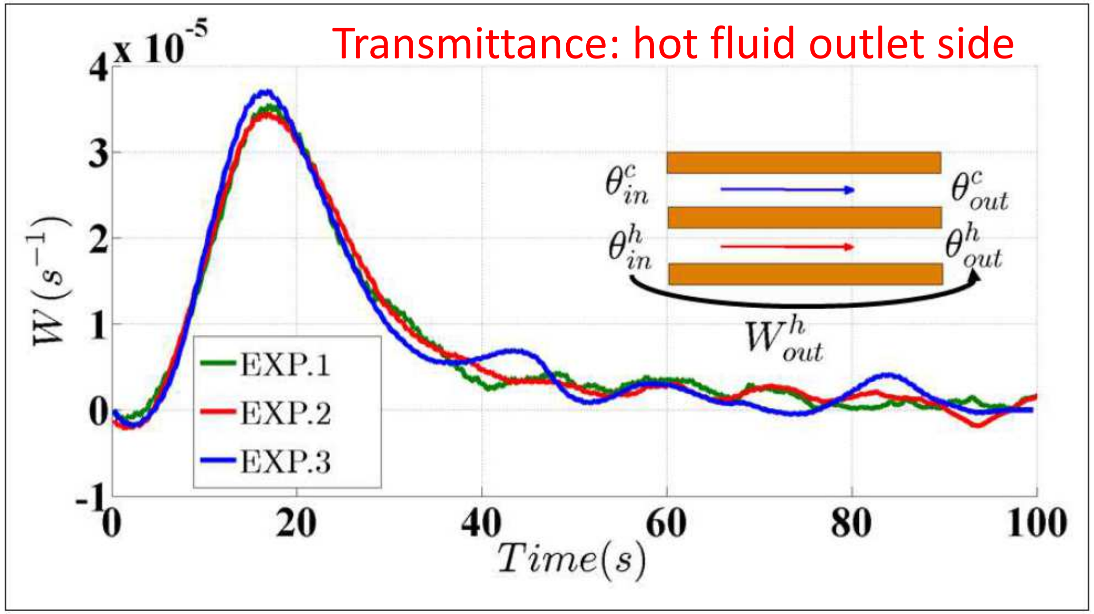


Inlet/outlet thermograms – Experiment 2



Inlet/outlet thermograms – Experiment 3

# Validation of the concept



Regularization: Tikhonov, 0 order

$$\hat{\boldsymbol{W}}^q_\mu = \mathrm{Arg}\left(\min_{\boldsymbol{W}}\left(\underbrace{\|\boldsymbol{r}(\boldsymbol{W})\|_2^2}_{\text{least squares sum}} + \underbrace{\mu^2\,\|\boldsymbol{W}\|_2^2}_{\text{penalisation}}\right)\right)$$

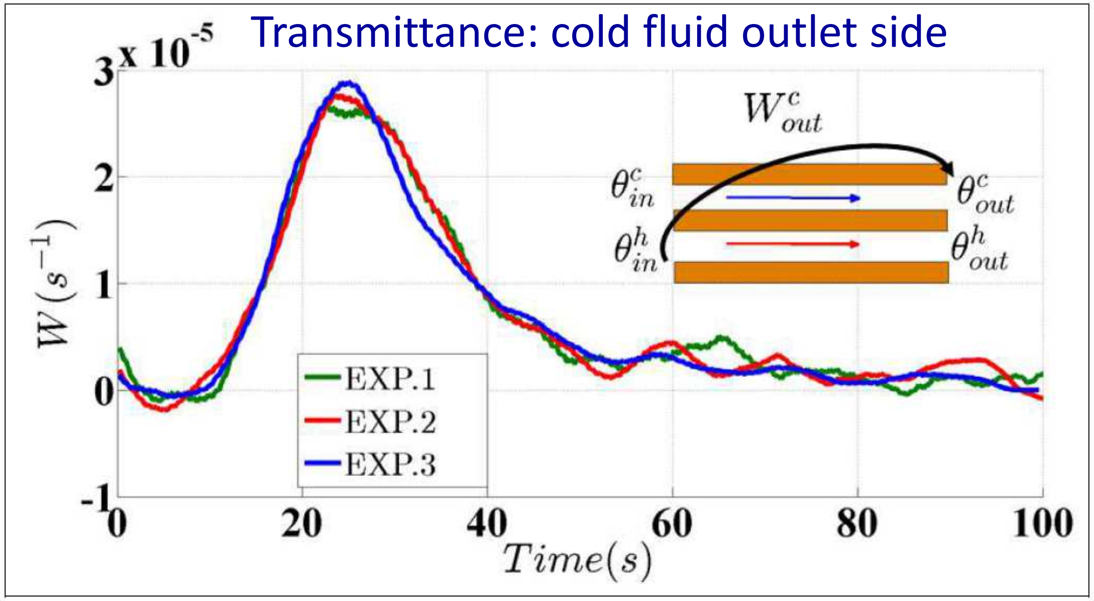
with $\boldsymbol{r}(\boldsymbol{W}) \equiv \boldsymbol{\theta}_q^{\exp} - \mathbf{M}(\boldsymbol{\theta}_1^{\exp})\,\boldsymbol{W}$

| $\mu$ (K.s) | $RMSR$ (K) | |
|---|---|---|
| | $\theta^c_{out}$ | $\theta^h_{out}$ |
| 1300 | 0.0050 | 0.0064 |
| 1800 | 0.0052 | 0.0060 |
| 1400 | 0.0052 | 0.0067 |

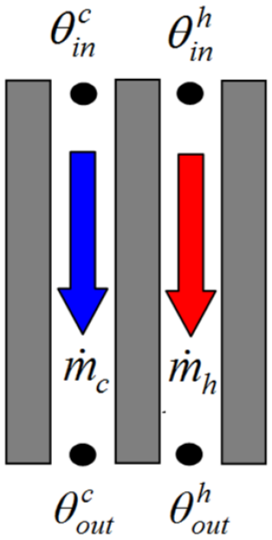
Choice of regularization hyperparameter $\mu$:
Morozov's criterion

$$RMRS\;(\mu) = \boldsymbol{r}\,(\hat{\boldsymbol{W}}^q_\mu)\;/\;\sqrt{m} \approx \sigma$$

$\hat{\sigma} \approx 0.0066$ K (before excitation)



# Assesment of the exchanger effectivness through time integration of transmittances

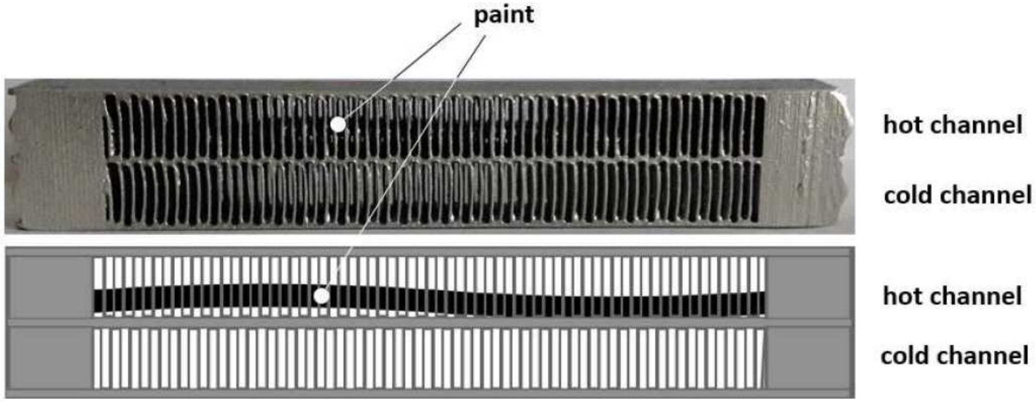


$$\varepsilon = \frac{Q_c}{Q_{max}} = \frac{C_c\,(\theta_{out}^{c,ss} - \theta_{in}^{c,ss})}{C_{min}\,(\theta_{in}^{h,ss} - \theta_{in}^{c,ss})} = \frac{\theta_{out}^{c,ss}}{\theta_{in}^{h,ss}} = W_{out}^{c,ss}$$

| | $W_{out}^{c,\,ss}$ | $W_{out}^{h,\,ss}$ |
|---|---|---|
| first experiment | 0.627 | 0.672 |
| second experiment | 0.619 | 0.674 |
| third experiment | 0.563 | 0.641 |

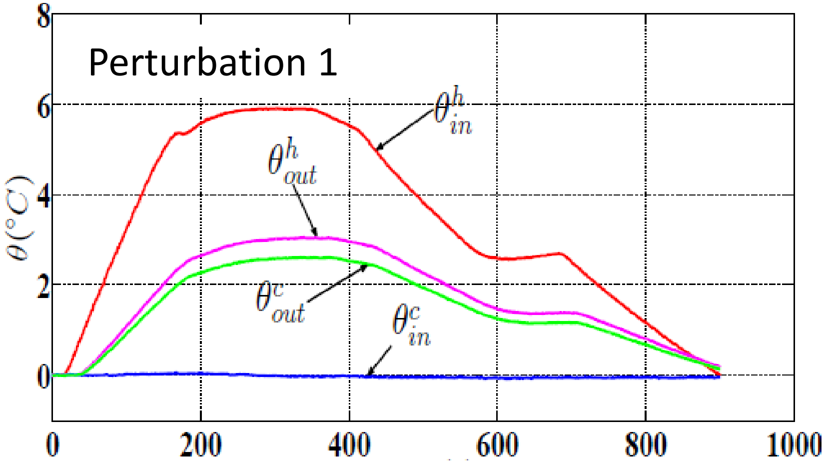
$$\varepsilon = \left( \frac{\theta_{out}^{c}\ (t \rightarrow +\infty)}{\theta_{in}^{h,ss}\ (t \rightarrow +\infty)} \right)_{\text{experiment 1}} = 0.6331$$

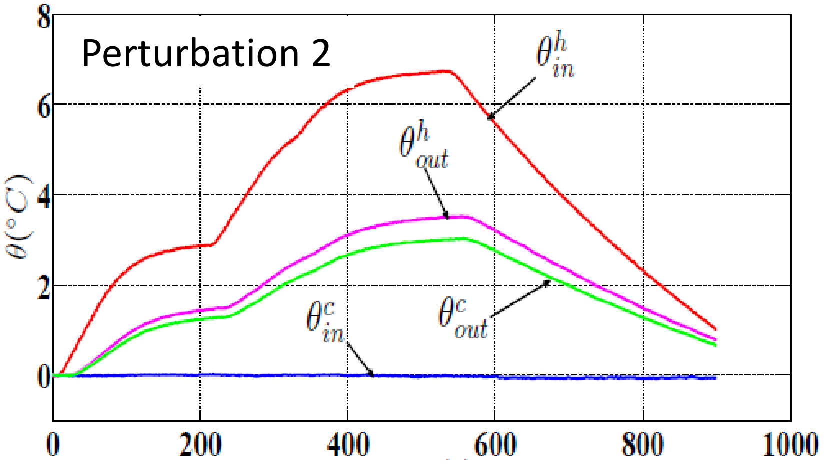
# Synthetic fouling of the plate fin exchanger and  non destructive testing



**with paint**

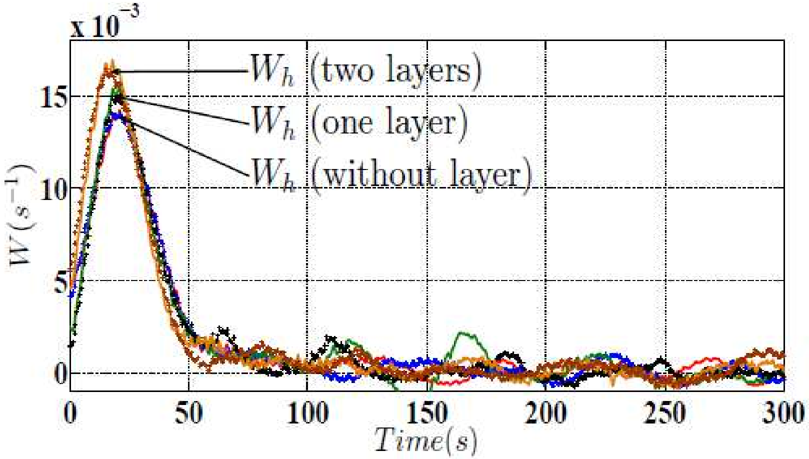
**without paint**



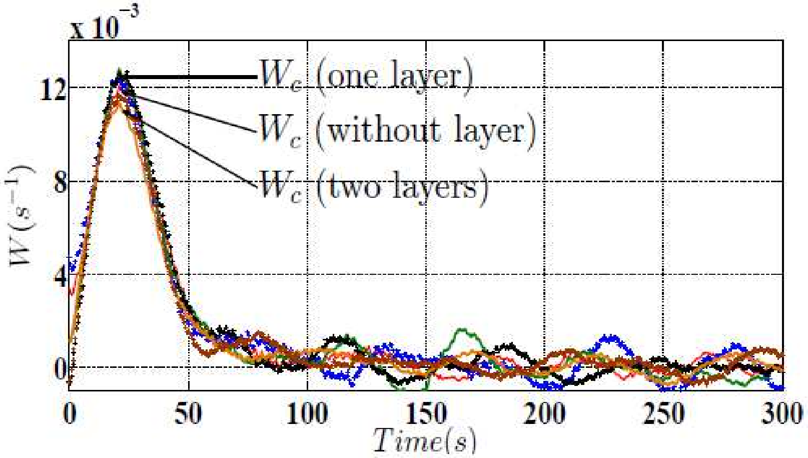


# Outlet transmittances

Hot fluid side



Cold fluid side



| Steady state effectiveness | Perturbation | $\dot{m}_h = \dot{m}_c/2 = 1$ kg / mn | | $\dot{m}_h = \dot{m}_c = 2$ kg / mn | |
|---|---|---|---|---|---|
| | | $\varepsilon_1 = (1 - W_h^{ss})$ | $\varepsilon_2 = 2W_c^{ss}$ | $\varepsilon_1 = (1 - W_h^{ss})$ | $\varepsilon_2 = W_c^{ss}$ |
| without fouling | 1 | 0.66 | 0.62 | 0.48 | 0.44 |
| | 2 | 0.65 | 0.62 | 0.47 | 0.45 |
| with fouling (one layer) | 1 | 0.66 | 0.62 | 0.48 | 0.45 |
| | 2 | 0.65 | 0.62 | 0.48 | 0.45 |
| with fouling (two layers) | 1 | 0.65 | 0.58 | 0.47 | 0.41 |
| | 2 | 0.64 | 0.58 | 0.46 | 0.41 |

# 8. Conclusions and perspectives

☐ Convolutive models in heat/mass transfer interesting for:

- modeling conjugated 3D heat transfer through model reduction:
  short-circuits non intrinsic Nusselt number-correlations in forced convection[6]
- experimental identification of Impulse Responses (IR):
  - Non Destructive Testing (NDT) of ageing of a model: from LTI to non LTI structure of model
  - design of virtual temperature or heat flux sensors
  - IR estimation easier if forced convection present: IR returns quickly to zero
  - however, need for a calibration

☐ Very large field of application:

- On-line characterization and NDT of heat exchangers[7] using steady state transmittances,
- Virtual sensor construction (calibration + inverse input problem) for radiation in furnaces[8] ,
- Pollutant source estimation[9] (inverse input problem in turbulent mass transfer),
- Management of heat storage.

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[6] A. Degiovanni, B. Rémy, An alternative to heat transfer coefficient: a relevant model of heat transfer between a developed fluid flow and a non-isothermal wall in the transient regime, *International Journal of Thermal Sciences* , Volume 102, April 2016, Pages 62–77.

[7] W. Al Hadad, V. Schick, D. Maillet, Fouling detection in a shell and tube heat exchanger using variation of its thermal impulse responses: Methodological approach and numerical verification, *Applied Thermal Engineering*, Volume 155 (2019) 612–619.

[8] Thomas Loussouarn, Denis Maillet, Benjamin Remy, Diane Dan, Model reduction for experimental thermal characterization of a holding furnace, *Heat and Mass Transfer*, Volume 54, Issue 8, 1 (2018), Pages 2443-2452, DOI 10.1007/s00231-017-2156-7.

[9] F. Chata, E. Belut, D. Maillet, F.X. Keller, A. Taniere, Estimation of an aerosol source in forced ventilation through prior identification of a convolutive model, *International Journal of Heat and Mass Transfer* 108 (2017) 1623–1633.

☐ Perspectives:

- optimal deconvolution, for minimizing estimation bias and standard deviation,
- in space domain: minimizing dependence of IR on the arbitrary type of BC at the frontier,
- in time domain, IR valid for linear forced response:
  - no relaxation of the initial temperature field (uniform or steady state),
  - otherwise relaxation caused by past excitations
- if relaxation term, possible use of AutoRegressive models with eXternal inputs[10] (ARX).

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[10] T. Loussouarn, D. Maillet, B. Rémy, V. Schick, D. Dan, Indirect measurement of temperature inside a furnace, ARX model identification, *Journal of Physics: Conference Series*, Volume 1047, Issue 1, 4 July 2018, Article number 012006 012, 9th International Conference on Inverse Problems in Engineering, ICIPE 2017; University of Waterloo; Canada; May 23-26, 2017 Code 137986

Thank you for your attention !